Algorithmic information, plane Kakeya sets, and conditional dimension

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Theoretical computer scientists are sometimes accused of insufficient attention to applications. In that spirit, this talk is devoted to an application of the theory of computing. The application is to pure mathematics!

We formulate the conditional Kolmogorov complexity of $x$ given $y$ at precision $r$, where $x$ and $y$ are points in Euclidean spaces and $r$ is a natural number. We demonstrate the utility of this notion in two ways.

1. We prove a point-to-set principle that enables one to use the (relativized, constructive) dimension of a single point in a set $E$ in a Euclidean space to establish a lower bound on the (classical) Hausdorff dimension of $E$. We then use this principle, together with conditional Kolmogorov complexity in Euclidean spaces, to give a new proof of the known, two-dimensional case of the Kakeya conjecture. This theorem of geometric measure theory, proved by Davies in 1971, says that every plane set containing a unit line segment in every direction has Hausdorff dimension 2.

2. We use conditional Kolmogorov complexity in Euclidean spaces to develop the lower and upper conditional dimensions $\dim(x|y)$ and $\Dim(x|y)$ of $x$ given $y$, where $x$ and $y$ are points in Euclidean spaces. Intuitively these are the lower and upper asymptotic algorithmic information densities of $x$ conditioned on the information in $y$. We prove that these conditional dimensions are robust and that they have the correct information-theoretic relationships with the well-studied dimensions $\dim(x)$ and $\Dim(x)$ and mutual dimensions $\mdim(x:y)$ and $\Mdim(x:y)$.

This is joint work with Neil Lutz.