Fuzzy modeling of conceptual spaces

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Abstract. It is introduced a new concept: H-sets which generalises the notion of fuzzy sets and is based on the concept of hemilattice. Then, we extend the attribute-value conceptual spaces over hemilattices, which gives us a kind of H-set, and prove some of their properties .

1 Introduction

Our goal is to define a structure for a space of real concepts $\mathcal{R}$ which should provide us a way to test some order relations such as subsumption and to operate generalisation and unification. Given two real concepts **a** and **b**, we denote $\mathbf{a} \leq \mathbf{b}$ and we read "**b** subsumes **a**" or "**a** is subsumed by **b**" if **a** is a more particular concept than **b** or, equivalently, **b** is a more general concept than **a**. By generalisation of two real concepts we denote the most particular concept more general than both of them; by unification we denote the most general concept more particular than the given concepts.

2 Hemilattices and Hemilatticeal Sets

In this section we will introduce a new concept which generalises the fuzzy sets. This is the notion of a set over a hemilattice. First we will present some preliminary definitions.

Definition 1. We say that $(H, \vee, \wedge)$ is a hemilattice iff

- $(H, \leq)$ is a partially ordered set
- $\forall a,b \in H \Rightarrow a \geq a \wedge b$ and $b \geq a \wedge b$
- $\forall a,b \in H \Rightarrow a \leq a \vee b$ and $b \leq a \vee b$.

Remark. Every lattice is a hemilattice.

Proposition 2. *If* $(H, \vee, \wedge, 0, 1, \leq)$ *is a hemilattice with* 0 *first element and* 1 *last element then* 0 *is idempotent relatively to* $\wedge$ *and* 1 *is idempotent relatively to* $\vee$.

Proof. Let $h \in H$, then $0 \wedge h \leq 0 \Rightarrow 0 \wedge h = 0$. Let $h \in H$, then $1 \vee h \geq 1 \Rightarrow 1 \vee h = 1$. □

Definition 3. Let $(H, \vee, \wedge)$ be a hemilattice, we say that it has the:

– c-property iff $\vee$ and $\wedge$ are commutative operations;
– a-property iff $\vee$ and $\wedge$ are associative operations;
– d-property iff $\vee$ and $\wedge$ are distributive operations each over the other;
– m-property iff $\forall x, y, z \in H,\ x \leq z \Rightarrow x \vee (y \wedge z) = (x \vee y) \wedge z$;
– 01-property iff 0 is neutral element for $\vee$ and 1 is neutral element for $\wedge$.

Definition 4. Let $(H, \vee, \wedge, 0, 1, \leq)$ be a hemilattice with 0 as first element and 1 as last element and let X be an ordinary set. We define a hemilatticeal set A as a function $f_A :\ X \to H$.

`Notation:` We denote the set of H-sets of X as

$$\mathcal{F}_X^H = \{f_A | f_A :\ X \to H\}$$

We denote the set of characterisic functions of ordinary sets as

$$\Xi_X = \{\chi_A | \chi_A :\ X \to \{0, 1\}\}$$

Remark. Obviously $\Xi_X \subseteq \mathcal{F}_X^H$, which means that the ordinary sets are a particular case of H-sets of X.

Definition 5. We introduce two operations on hemilatticeal sets:

– Intersection: $(f_A \cap f_B)(x) = f_A(x) \wedge f_B(x)\ \ \forall x \in X$;
– Union: $(f_A \cup f_B)(x) = f_A(x) \vee f_B(x)\ \ \forall x \in X$;

Remark. The sets $\emptyset$ and X could be described as follows:

– $\emptyset$: $f_\emptyset(x) = 0$, $\forall x \in X$;
– X: $f_X(x) = 1$, $\forall x \in X$.

Definition 6. The set of hemilatticeal sets can be ordered as follows:

$$f_A \leq f_B \Leftrightarrow \forall x \in X\ f_A(x) \leq f_B(x)$$

Proposition 7. *If $(H, \vee, \wedge, 0, 1, \leq)$ is a hemilattice with 0 as first element and 1 as last element then $(\mathcal{F}_X^H, \cup, \cap, f_\emptyset, f_X, \leq)$ is a hemilattice with $f_\emptyset$ as first element and f_X as last element.*

Proof. Obviously $(\mathcal{F}, \subseteq)$ is partially ordered.
Let $f_A, f_B \in \mathcal{F}_X^H$; let $x \in X$ then
$(f_A \cap f_B)(x) = f_A(x) \wedge f_B(x) \leq f_A(x), \forall x \in X \Rightarrow f_A \cap f_B \leq f_A$ and
$(f_A \cap f_B)(x) = f_A(x) \wedge f_B(x) \leq f_B(x), \forall x \in X \Rightarrow f_A \cap f_B \leq f_B$.
Let $f_A, f_B \in \mathcal{F}_X^H$; let $x \in X$; then
$(f_A \cup f_B)(x) = f_A(x) \vee f_B(x) \geq f_A(x), \forall x \in X \Rightarrow f_A \cup f_B \geq f_A$ and
$(f_A \cup f_B)(x) = f_A(x) \vee f_B(x) \geq f_B(x), \forall x \in X \Rightarrow f_A \cup f_B \geq f_B$.
Let $f_A \in \mathcal{F}_X^H$; let $x \in X$ then $f_\emptyset(x) = 0 \leq f_A(x) \Rightarrow f_\emptyset \leq f_A$.
Let $f_A \in \mathcal{F}_X^H$; let $x \in X$; then $f_X(x) = 1 \geq f_A(x) \Rightarrow f_X \geq f_A$ □

Proposition 8. *If* $(H, \vee, \wedge, 0, 1, \leq)$ *is a lattice with* 0 *as first element and* 1 *as last element then* $(\mathcal{F}_X^H, \cup, \cap, f_\emptyset, f_X, \leq)$ *is a lattice with* $f_\emptyset$ *as first element and* f_X *as last element.*

Proof. Let $f_C \in \mathcal{F}_X^H$ so that $f_C \leq f_A$, $f_C \leq f_B$ and $f_C \geq f_A \cap f_B$ then

$$f_C \leq f_A \;\Rightarrow\; \forall x \in X \;\; f_C(x) \leq f_A(x)$$

$$f_C \leq f_B \;\Rightarrow\; \forall x \in X \;\; f_C(x) \leq f_B(x)$$

$$f_C \geq f_A \cap f_B \;\Rightarrow\; \forall x \in X \;\; f_C(x) \geq f_A(x) \wedge f_B(x)$$

From the precendind three statements and from the fact that H is a lattice it follows that $\forall x \in X \;\; f_C(x) = f_A(x) \wedge f_B(x)$ which implies that $f_C = f_A \cap f_B$. Let $f_C \in \mathcal{F}_X^H$ so that $f_C \geq f_A$, $f_C \geq f_B$ and $f_C \leq f_A \cup f_B$ then

$$f_C \geq f_A \;\Rightarrow\; \forall x \in X \;\; f_C(x) \geq f_A(x)$$

$$f_C \geq f_B \;\Rightarrow\; \forall x \in X \;\; f_C(x) \geq f_B(x)$$

$$f_C \leq f_A \cup f_B \;\Rightarrow\; \forall x \in X \;\; f_C(x) \leq f_A(x) \vee f_B(x)$$

From the precendind three statements and from the fact that H is a lattice it follows that $\forall x \in X \;\; f_C(x) = f_A(x) \vee f_B(x)$ which implies that $f_C = f_A \cup f_B$ □

Remark. If $(H, \vee, \wedge, 0, 1, \leq)$ is a (hemi)lattice with 0 as first element and 1 as last element has any of a, c, d, m, 01-properties then $(\mathcal{F}_X^H, \cup, \cap, f_\emptyset, f_X, \leq)$ is a (hemi)lattice with a, c, d, m, 01-property, respectively. Proof is omited because it is obvious.

Examples of Hemilattices:

0. The *singular hemilattice* $(\{\#\}, \mathbf{max}, \mathbf{min}, 0, 1, \leq)$ where **max**, **min**, 0, 1, and $\leq$ have the usual meaning is obviously a hemilattice, because it is a lattice.

1. The *boolean hemilattice* $(\{0, 1\}, \mathbf{max}, \mathbf{min}, 0, 1, \leq)$ where **max**, **min**, 0, 1, and $\leq$ have the usual meaning is obviously a hemilattice, because it is a lattice.

2. The *probability hemilattice*: $([0, 1], \oplus, \cdot, 0, 1, \leq)$; $\cdot$ is the usual multiplication, $\leq$ is the usual order relation and $\oplus$ is defined by

$$a \oplus b = a + b - ab$$

Proposition 9. $\oplus$ *is well defined.*

Proof. $\oplus$ is well defined iff $0 \leq a \oplus b \leq 1$.
$0 \leq a \oplus b \Leftrightarrow 0 \leq a + b - ab \Leftrightarrow 0 \leq a + b(1 - a)$ which is true, because $a \geq 0$, $b \geq 0$ and $a \leq 1 \Rightarrow (1 - a) \geq 0$.
$a \oplus b \leq 1 \Leftrightarrow a + b - ab \leq 1 \Leftrightarrow 0 \leq 1 - a - b + ab \Leftrightarrow 0 \leq (1 - a)(1 - b)$ which is true, because $a \leq 1 \Rightarrow (1 - a) \geq 0$ and $b \leq 1 \Rightarrow (1 - b) \geq 0$. □

Proposition 10. $([0,1], \oplus, \cdot, 0, 1, \leq)$ *is a hemilattice.*

Proof. $([0,1], \oplus, \cdot, 0, 1, \leq)$ is a hemilattice iff $([0,1], \leq)$ is partially ordered, $ab \leq a$, $ab \leq b$, $a \oplus b \geq a$ and $a \oplus b \geq b$.
$([0,1], \leq)$ is obviously partially ordered because it is totally ordered.
$b \leq 1$ and $a \geq 0 \Rightarrow ab \leq a$;
$a \leq 1$ and $b \geq 0 \Rightarrow ab \leq b$;
$a \oplus b \geq a \Leftrightarrow a + b - ab \geq a \Leftrightarrow b - ab \geq 0 \Leftrightarrow b(1-a) \geq 0$, which is true;
$a \oplus b \geq b \Leftrightarrow a + b - ab \geq b \Leftrightarrow a - ab \geq 0 \Leftrightarrow a(1-b) \geq 0$, which is true. □

Remark. The probability hemilattice is not a lattice, because $0.5 \cdot 0.5 = 0.25 \neq 0.5$; $0.5 \oplus 0.5 = 0.75 \neq 0.5$

3. The *fuzzy hemilattice* $([0,1], \mathbf{max}, \mathbf{min}, 0, 1, \leq)$ is obviously a hemilattice, because it is a lattice.

4. The *trust intervals hemilattice* $(\mathcal{I}_{[0,1]}, \overline{\cup}, \cap, \subseteq)$ where $\mathcal{I}_{[0,1]} = \{[a,b] | 0 \leq a \leq b \leq 1\}$, **max**, **min** and $\cap$ have the usual meaning and $\overline{\cup}$ is defined by $[a_1, b_1] \overline{\cup} [a_2, b_2] = [\mathbf{min}\{a_1, a_2\}, \mathbf{max}\{b_1, b_2\}]$ is obviously a lattice, thus it is a hemilattice.

3 Conceptual Spaces over Hemilattices

In this section we will introduce the notion of conceptual spaces over hemilattices by naturally generalising the bare conceptual spaces, which are a particular instance of conceptual spaces over hemilattices, i. e. they are conceptual spaces over the boolean (hemi)lattice. The main result of this section is Theorem 8 stating that the conceptual space over a hemilattice is a hemilattice that preserves a, c and 01-properties and does not preserve d and m-properties of the initial hemilattice.

Definition 1. An attribute-value conceptual space is a structure

$$\mathcal{C} = (A, \{V_a\}_{a \in A})$$

where A is a set of attributes and for every attribute $a \in A$, $V_a \neq \emptyset$ is a set of atomic values that attribute a can take. An element $m \in \mathcal{C}$ is a set

$$m = \{(a, v) | a \in A, v \in V_a\}$$

Example : A={size,colour} V_{size}={small,medium,big} V_{colour}={red,green, yellow,brown} the concept of apple would be represented as apple ={(size,medium), (colour,red),(colour,yellow),(colour,green)} and the concept of red-apple as red-apple={(size,medium),(colour,red)}.

Definition 2. Let $(H, \vee, \wedge, 0, 1, \leq)$ be a hemilattice with first and last element. We define the H-conceptual space over the hemilattice H as

$$\mathcal{C}^H = (A, \{V_a\}_{a \in A}, H)$$

An element of the H-conceptual space $\mathcal{C}^H$ is a set

$$m = \{(a, v, h) | a \in A, v \in V_a, h \in H\}$$

such that $\forall (a, v, h), (a', v', h') \in m, a = a', v = v' \Rightarrow h = h'$.

Definition 3. We introduce $\mathrm{pr}_a : \mathcal{C}^H \to \mathcal{F}^H_{V_a}$,

$$\mathrm{pr}_a(m)(v) = h \Leftrightarrow (a, v, h) \in m$$

called the projection on attribute a.

Definition 4. We introduce $\mathrm{pr}_{a,v} : \mathcal{C}^H \to H$,

$$\mathrm{pr}_{a,v}(m) = h \Leftrightarrow (a, v, h) \in m$$

called the projection on attribute-value a, v.

Remark. $\forall m \in \mathcal{C}^H \; m = \bigcup_{a \in A} \bigcup_{v \in V_a} \{a\} \times \{v\} \times \mathrm{pr}_{a,v}(m) = \bigcup_{a \in A} \times (V_a, pr_a(V_a))$.

Remark. We observe that

$$\mathcal{C}^H \simeq \mathcal{F}^H_{\bigcup\limits_{a \in A} \{a\} \times V_a}$$

and the following structure: $(\mathcal{C}^H, \cup, \cap, f_\emptyset, f_{\bigcup\limits_{a \in A} \{a\} \times V_a}, \subseteq)$ is a (hemi)lattice with first and last element if H is and will preserve the eventual properties of H.

`Notation`: We denote:

- the void concept: $\oslash = f_\emptyset$;
- the most general concept: $\Box = f_{\bigcup\limits_{a \in A} \{a\} \times V_a}$

Definition 5. We define $\Psi : \mathcal{C}^H \to \mathcal{C}^H$ as follows

$$\Psi(\alpha) = \begin{cases} \alpha \; if \; \forall a \in A \; pr_a(\alpha) \neq f^a_\emptyset \\ f_\emptyset \; if \; \exists a \in A \; pr_a(\alpha) = f^a_\emptyset \end{cases}$$

`Notation`: We will denote $\mathcal{C}^H_\emptyset = Im\Psi(\mathcal{C}^H) \subseteq \mathcal{C}^H$.

Definition 6. We introduce two operations on $\mathcal{C}^H_\emptyset$:

- Unification: $\alpha \mathcal{U} \beta = \Psi(\alpha \cap \beta) \quad \forall \alpha, \beta \in \mathcal{C}^H_\emptyset$
- Generalization: $\alpha \mathcal{G} \beta = \alpha \cup \beta \quad \forall \alpha, \beta \in \mathcal{C}^H_\emptyset$

Proposition 7. *If H is a hemilattice with 01-property then :*

- $\Box\mathcal{U}\alpha = \alpha\mathcal{U}\Box = \alpha \ \ \forall\alpha \in \mathcal{C}_{\emptyset}^{H}$
- $\oslash\mathcal{U}\alpha = \alpha\mathcal{U}\oslash = \oslash \ \ \forall\alpha \in \mathcal{C}_{\emptyset}^{H}$
- $\Box\mathcal{G}\alpha = \alpha\mathcal{G}\Box = \Box \ \ \forall\alpha \in \mathcal{C}_{\emptyset}^{H}$
- $\oslash\mathcal{G}\alpha = \alpha\mathcal{G}\oslash = \alpha \ \ \forall\alpha \in \mathcal{C}_{\emptyset}^{H}$

Proof.

$$\Box\mathcal{U}\alpha = \Psi(\Box \cap \alpha) = \Psi(\alpha) = \alpha$$

$$\oslash\mathcal{U}\alpha = \Psi(\alpha \cap \oslash) = \Psi(\oslash) = \oslash$$

$$\Box\mathcal{G}\alpha = \alpha \cup \Box = \Box$$

$$\oslash\mathcal{G}\alpha = \alpha \cup \oslash = \alpha$$

□

Theorem 8. *If $(H, \vee, \wedge, 0, 1, \leq)$ is a (hemi)latice with 0 first element and 1 last element then*

$$(\mathcal{C}_{\emptyset}^{H}, \mathcal{G}, \mathcal{U}, \oslash, \Box, \leq)$$

is a (hemi)lattice with $\oslash$ first and $\Box$ last element.

Proof. Obviously $\oslash$ and $\Box$ are first respectively last element. Obviously $\mathcal{G}$ has the requested properties if H is a hemilattice and is a supremum if H is lattice. All we have to prove now is about $\mathcal{U}$.
Let $\alpha, \beta \ \in \ \mathcal{C}_{\emptyset}^{H}$ then $\alpha\mathcal{U}\beta = \Psi(\alpha \cap \beta) =$

$$\begin{cases} \alpha \cap \beta \, if \quad \forall a \in A \; pr_a(\alpha \cap \beta) \neq f_{\emptyset}^{a} \\ \quad \oslash \quad otherwise \end{cases}$$

which is obviously enclosed in α and β.
For the case when H is lattice let $\alpha, \beta, \gamma \ \in \ \mathcal{C}_{\emptyset}^{H}$ so that $\gamma \leq \alpha$, $\gamma \leq \beta$ and $\gamma \geq \alpha\mathcal{U}\beta$ then from the first two results that $\gamma \subseteq \alpha \cap \beta$. we have two cases:
a) $\forall \ a \in A$; $pr_a(\alpha \cap \beta) \neq f_{\emptyset}^{a}$ then $\alpha\mathcal{U}\beta = \alpha \cap \beta$ and results $\gamma \supseteq \alpha \cap \beta$ q.e.d.
b) $\exists \ a \in A$; $pr_a(\alpha \cap \beta) = f_{\emptyset}^{a}$ then $\alpha\mathcal{U}\beta = f_{\emptyset} = \oslash$ but $\gamma \subseteq \alpha \cap \beta \ \Rightarrow \ \exists \ a \in A \ pr_a(\gamma) = f_{\emptyset}^{a}$ but $\gamma \in \mathcal{C}_{\emptyset}^{a} \ \Rightarrow \ \gamma = f_{\emptyset} = \oslash$ q.e.d. □

Proposition 9. *If H is as above and has c, a and 01-properties then $\mathcal{C}_{\emptyset}^{H}$ has c, a and 01-properties.*

Proof. The a and c-properties of generalisation results from the the same properties of $\cup$.The 01-property results from 7. All we have to prove now is the c and a-property of unification.
c-property: $\alpha\mathcal{U}\beta = \Psi(\alpha \cap \beta) = \Psi(\beta \cap \alpha) = \Psi(\beta\mathcal{U}\alpha)$.
a-property:$(\alpha\mathcal{U}(\beta\mathcal{U}\gamma)) = (\alpha\mathcal{U}\Psi(\beta \cap \gamma)) = \Psi(\alpha \cap \Psi(\beta \cap \gamma)) =$

$$\begin{cases} \Psi(\alpha \cap (\beta \cap \gamma)) \, if \quad \forall a \in A \; pr_a(\beta \cap \gamma) \neq f_{\emptyset}^{a} \\ \qquad \oslash \qquad otherwise \end{cases}$$

$$\begin{cases} \alpha \cap \beta \cap \gamma \; if \quad \forall a \in A \; pr_a(\beta \cap \gamma) \neq f^a_\emptyset \quad and \quad pr_a(\alpha \cap \beta \cap \gamma) \neq f^a_\emptyset \\ \quad \oslash \qquad otherwise \end{cases}$$

But $pr_a(\alpha \cap \beta \cap \gamma) \neq f^a_\emptyset \Rightarrow pr_a(\beta \cap \gamma) \neq f^a_\emptyset$

$$\begin{cases} \alpha \cap \beta \cap \gamma \; if \quad \forall a \in A \; pr_a(\alpha \cap \beta \cap \gamma) \neq f^a_\emptyset \\ \quad \oslash \qquad otherwise \end{cases}$$

In an analoguous way we get the right hand formula. □

Remark. If H is as above and has d and m-properties then $\mathcal{C}^H_\emptyset$ does not have d and m-properties.

Counterexample

Let H be the boolean hemilatice.In the following counterexamples we will not emphasize the 3-uples with 0 on the third position and we will replace a 3-uples of the form (a,v,1) with (a,v) obtaining a conceptual space isomorph with the boolean one but wich facilitate the calculus.
1. Distributivity $\mathcal{C}^H = (\{1,2\}, \{\{a,b,c,d\}, \{a,b,c,d\}\}, H)$
a) $\mathcal{G}$ over $\mathcal{U}$

$$\alpha = \{(1,c),(2,d)\}, \beta = \{(1,a),(2,a)\}, \gamma = \{(1,a),(2,b)\}$$

$$\alpha\mathcal{G}(\beta\mathcal{U}\gamma) = (1,c),(2,d)$$

$$(\alpha\mathcal{G}\beta)\mathcal{U}(\alpha\mathcal{G}\gamma) = (1,c),(1,a),(2,d) \neq (1,c),(2,d)$$

b) $\mathcal{U}$ over $\mathcal{G}$

$$\alpha = \{(1,c),(2,d)\}, \beta = \{(1,a),(2,d)\}, \gamma = \{(1,c),(2,b)\}$$

$$\alpha\mathcal{U}(\beta\mathcal{G}\gamma) = (1,c),(2,d)$$

$$(\alpha\mathcal{U}\beta)\mathcal{G}(\alpha\mathcal{U}\gamma) = \emptyset \neq (1,c),(2,d)$$

2. Modularity $\mathcal{C}^H = (\{1,2\}, \{\{a,b,c\}, \{a,b,c\}\}, H)$

$$\alpha = \{(1,b),(2,c)\}, \beta = \{(1,a),(2,b)\}, \gamma = \{(1,b),(2,b),(2,c)\}$$

$$\alpha \leq \gamma$$

$$\alpha\mathcal{G}(\beta\mathcal{U}\gamma) = (1,b),(2,c)$$

$$(\alpha\mathcal{G}\beta)\mathcal{U}\gamma) = (1,b),(2,b),(2,c)$$

Definition 10. Let $\mathcal{R}$ be a space of real concepts. We say that $\mathcal{C}_H$ is an attribute-value decomposition of $\mathcal{R}$ if there is an injective moprphism $\Phi \;\; \mathcal{R} \rightarrow \mathcal{C}_H$ which perserves the order and operations.

4 Further Extensions

The H-conceptual spaces described above could be extended by allowing the attributes to take composed values, meaning values that could be themself concepts. With this improvement the new concepts could be represented like trees, dags, or graphs. But this is not a problem because we can represent any graph as the list of his edges.

Being more explicit if we have a graph with weighted edges (as a general concept wold look like) and the edge between nodes n_i and n_j has the weight $w_{i,j}$ then we would have an attribute in the concept named $n_i n_j$ with the value $w_{i,j}$ so we would be able to use Theorem 8.

5 Concluding Remarks

By introducing this new concepts of hemilattice and conceptual spaces over hemilattices we are now able to treat problems that uses conceptual spaces such as the learnig algorithms ID3,C4.5 or Tensor product varible binding in a new fuzzyfied way and to improve their resemblance with real spaces of concepts.

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