

Resource-Bounded Strong Dimension versus Resource-Bounded Category

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Abstract

Classically it is known that any set with packing dimension less than 1 is meager in the sense of Baire category. We establish a resource-bounded extension: if a class X has Δ -strong dimension less than 1, then X is Δ -meager. This has the applications of explaining some of Lutz's simultaneous Δ -meager, Δ -measure 0 results and providing a new proof of a Gu's strong dimension result on infinitely-often classes.

Key words: computational complexity, resource-bounded category, resource-bounded dimension, resource-bounded measure

1 Introduction

The most common mathematical notions of size and dimension now have resource-bounded versions that are useful for complexity classes. We use Δ to denote a resource bound such as p (polynomial time) or p space (polynomial space).

- *Resource-Bounded Category* [1]: Extension of *Baire category*. Complexity classes may be Δ -meager or Δ -comeager (or neither).
- *Resource-Bounded Measure* [2]: Extension of *Lebesgue measure*. The Δ -measure of a complexity class X is denoted $\mu_\Delta(X)$. A class X may have $\mu_\Delta(X) = 0$ or $\mu_\Delta(X) = 1$ (or neither, in which case the class is called not Δ -measurable).

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- *Resource-Bounded Dimension* [3]: Extension of *Hausdorff dimension* [4]. Each complexity class X has a Δ -dimension $\dim_{\Delta}(X)$ that is always a real number in $[0,1]$.
- *Resource-Bounded Strong Dimension* [5]: Extension of *packing dimension* [6,7]. Each complexity class X has a Δ -strong dimension $\text{Dim}_{\Delta}(X)$ that is always a real number in $[0,1]$.

In general, resource-bounded category and resource-bounded measure are incomparable: Δ -meager does not imply Δ -measure 0, and vice versa. Regarding measure versus the two notions of dimension, the following hold for every class X :

$$\dim_{\Delta}(X) \leq \text{Dim}_{\Delta}(X)$$

and

$$\dim_{\Delta}(X) < 1 \Rightarrow \mu_{\Delta}(X) = 0.$$

In particular, it follows that if the Δ -strong dimension of X is less than 1, then X has Δ -measure 0. We show that $\text{Dim}_{\Delta}(X) < 1$ also implies X is Δ -meager. This is an extension of the analogous relationship between packing dimension and Baire category (see Edgar [8]).

We give two applications of this result:

- An explanation of why some complexity classes in the work of Lutz [1] have Δ -measure 0 and are also Δ -meager. It is because they have Δ -strong dimension less than 1 (Gu [9]).
- A new category-based proof of Gu's result regarding the strong dimension of infinitely-often classes [9].

Section 2 contains preliminaries and background on category, measure, and dimension. Our main theorem is presented in section 3. The applications are given in section 4.

2 Category, Measure, and Dimension

The *Cantor space* \mathbf{C} is the set of all infinite binary sequences. A *language* (or *decision problem*) is a subset of $\{0,1\}^*$. We identify each language with the element of Cantor space that is its characteristic sequence according to the standard enumeration of $\{0,1\}^*$. In this way, complexity classes (sets of languages) are viewed as subsets of Cantor space.

A *constructor* is a function $\delta : \{0,1\}^* \rightarrow \{0,1\}^*$. The *result* of a constructor is the unique sequence $R(\delta) \in \mathbf{C}$ that extends $\delta^{(n)}(\lambda)$ for all n . (Here λ is the empty string.)

Throughout this paper, Δ denotes a *resource bound* [2]. Examples of Δ include:

$$\begin{aligned} \text{all} &= \{f \mid f : \{0, 1\}^* \rightarrow \{0, 1\}^*\} \\ \text{p} &= \{f \mid f \text{ is polynomial-time computable}\} \\ \text{p}_2 &= \{f \mid f \text{ is quasipolynomial-time computable}\} \\ \text{pspace} &= \{f \mid f \text{ is polynomial-space computable}\} \\ \text{comp} &= \{f \mid f \text{ is computable}\} \end{aligned}$$

For a resource bound Δ , we define the class

$$R(\Delta) = \{R(\delta) \mid \delta \in \Delta \text{ is a constructor}\}.$$

Then $R(\text{all}) = \mathbf{C}$, $R(\text{p}) = \mathbf{E}$, $R(\text{p}_2) = \mathbf{EXP}$, $R(\text{pspace}) = \mathbf{ESPACE}$, and $R(\text{comp}) = \mathbf{DEC}$. Each resource bound Δ yields notions of resourced-bounded category, measure, and dimension that work within the class $R(\Delta)$. We now review these concepts.

2.1 Category

Baire category classifies sets into two types: *first category* and *second category*. First category sets are also commonly called *meager*. A set is meager if it is a countable union of nowhere dense sets. An equivalent definition comes from Banach-Mazur games.

Let $X \subseteq \mathbf{C}$ and let Γ_I and Γ_{II} be two classes of functions. In the *Banach-Mazur game* $G[X; \Gamma_I, \Gamma_{II}]$ there are two players I and II. A *strategy* in the game is a constructor. In a play of the game, player I chooses a strategy $g \in \Gamma_I$ and player II chooses a strategy $h \in \Gamma_{II}$. The *result* of this play is the sequence $R(g, h) = R(h \circ g)$. Intuitively, the result is the sequence obtained when the two players start with the empty string and take turns extending it with their strategies. A *winning strategy* for player II is a strategy $h \in \Gamma_{II}$ such that for every $g \in \Gamma_I$, $R(g, h) \notin X$.

Theorem 2.1 (Banach and Mazur) *A class $X \subseteq \mathbf{C}$ is meager if and only if player II has a winning strategy in the game $G[X; \text{all}, \text{all}]$.*

Resource-bounded category [1] is defined by requiring player II's winning strategy to be computable within a resource bound.

Definition Let $X \subseteq \mathbf{C}$.

- (1) X is Δ -*meager* if player II has a winning strategy in the game $G[X; \text{all}, \Delta]$.
- (2) X is Δ -*comeager* if X^c is Δ -meager.
- (3) X is *meager in $R(\Delta)$* if $X \cap R(\Delta)$ is Δ -meager.

(4) X is *comeager* in $R(\Delta)$ if X^c is meager in $R(\Delta)$.

The *resource-bounded Baire category theorem* [1] tells us that $R(\Delta)$ is not Δ -meager.

2.2 Measure

A *martingale* is a function $d : \{0, 1\}^* \rightarrow [0, \infty)$ satisfying the averaging condition

$$d(w) = \frac{d(w0) + d(w1)}{2}$$

for all $w \in \{0, 1\}^*$. We say d *succeeds* on a sequence $S \in \mathbf{C}$ if

$$\limsup_{n \rightarrow \infty} d(S \upharpoonright n) = \infty.$$

(Here $S \upharpoonright n$ is the length n prefix of S .) The *success set* of d is

$$S^\infty[d] = \{S \in \mathbf{C} \mid d \text{ succeeds on } S\}.$$

Ville used martingales to give an equivalent definition of Lebesgue measure 0.

Theorem 2.2 (Ville [10]) *A class $X \subseteq \mathbf{C}$ has Lebesgue measure 0 if and only if there is a martingale d with $X \subseteq S^\infty[d]$.*

Resource-bounded measure [2] arises from putting resource bounds on the martingales. We say that $d : \{0, 1\}^* \rightarrow [0, \infty)$ is Δ -*computable* if there is an approximation $\hat{d} : \mathbb{N} \times \{0, 1\}^* \rightarrow \mathbb{Q}$ such that $|\hat{d}(r, w) - d(w)| \leq 2^{-r}$ for all $r \in \mathbb{N}, w \in \{0, 1\}^*$ and $\hat{d} \in \Delta$ (with r encoded in unary and the outputs encoded in binary).

Definition Let $X \subseteq \mathbf{C}$.

- (1) X has Δ -*measure 0*, written $\mu_\Delta(X) = 0$, if there is a Δ -computable martingale d with $X \subseteq S^\infty[d]$.
- (2) X has Δ -*measure 1*, written $\mu_\Delta(X) = 1$, if $\mu_\Delta(X^c) = 0$.
- (3) X has *measure 0 in $R(\Delta)$* , written $\mu(X \mid R(\Delta)) = 0$, if $\mu_\Delta(X \cap R(\Delta)) = 0$.
- (4) X has *measure 1 in $R(\Delta)$* , written $\mu(X \mid R(\Delta)) = 1$, if $\mu_\Delta(X^c \mid R(\Delta)) = 0$.

The *resource-bounded measure conservation theorem* [2] tells us that $R(\Delta)$ does not have Δ -measure 0.

2.3 Dimension and Strong Dimension

The most commonly used fractal dimension is the *Hausdorff dimension* $\dim_{\text{H}}(X)$. Lutz used success sets of functions called gales to characterize Hausdorff dimension. Let $s \geq 0$ be a real number. An s -gale is a function $d : \{0, 1\}^* \rightarrow [0, \infty)$ satisfying the condition

$$d(w) = \frac{d(w0) + d(w1)}{2^s}$$

for all $w \in \{0, 1\}^*$. Note that a martingale is a 1-gale. “Succeeds on” and “success set” are defined for s -gales in the same way as for martingales.

Theorem 2.3 (Lutz [3]) *For every $X \subseteq \mathbf{C}$,*

$$\dim_{\text{H}}(X) = \inf \{s \mid \text{there is an } s\text{-gale } d \text{ with } X \subseteq S^\infty[d]\}.$$

Another common fractal dimension is the *packing dimension* $\dim_{\text{P}}(X)$. This has an analogous gale characterization using the notion of strong success. An s -gale d *succeeds strongly* on a sequence $S \in \mathbf{C}$ if

$$\liminf_{n \rightarrow \infty} d(S \upharpoonright n) = \infty.$$

The *strong success set* of d is

$$S_{\text{str}}^\infty[d] = \{S \in \mathbf{C} \mid d \text{ succeeds strongly on } S\}.$$

Theorem 2.4 (Athreya, Hitchcock, Lutz, and Mayordomo [5]) *For every $X \subseteq \mathbf{C}$,*

$$\dim_{\text{P}}(X) = \inf \{s \mid \text{there is an } s\text{-gale } d \text{ with } X \subseteq S_{\text{str}}^\infty[d]\}.$$

Based on Theorems 2.3 and 2.4, resource-bounded dimension and resource-bounded strong dimension are defined as extensions of Hausdorff dimension and packing dimension, respectively, by requiring the gales to be computable within a resource bound.

Definition Let $X \subseteq \mathbf{C}$.

- (1) The Δ -dimension of X is

$$\dim_{\Delta}(X) = \inf \{s \mid \text{there is a } \Delta\text{-computable } s\text{-gale } d \text{ with } X \subseteq S^\infty[d]\}.$$

- (2) The Δ -strong dimension of X is

$$\text{Dim}_{\Delta}(X) = \inf \{s \mid \text{there is a } \Delta\text{-computable } s\text{-gale } d \text{ with } X \subseteq S_{\text{str}}^\infty[d]\}.$$

- (3) The *dimension of X in $R(\Delta)$* is $\dim(X \mid R(\Delta)) = \dim_{\Delta}(X \cap R(\Delta))$.
- (4) The *strong dimension of X in $R(\Delta)$* is $\text{Dim}(X \mid R(\Delta)) = \text{Dim}_{\Delta}(X \cap R(\Delta))$.

We say that an s -gale d is *exactly Δ -computable* if the range of d is rational and the values can be computed by a function in Δ . The *exact computation lemma* [3] tells us that we may restrict to exactly computable s -gales in the above definitions.

Proposition 2.5 ([3,5]) *Let $X \subseteq \mathbf{C}$.*

- (1) $0 \leq \dim_{\Delta}(X) \leq \text{Dim}_{\Delta}(X) \leq 1$.
- (2) *If $\dim_{\Delta}(X) < 1$, then X has Δ -measure 0.*

3 Main Theorem

It is known classically that if the packing dimension $\dim_{\mathbf{P}}(X) < 1$, then X is meager (see Edgar [8, page 65]). We now establish the resource-bounded extension of this fact.

Theorem 3.1 *Let $X \subseteq \mathbf{C}$.*

- (1) *If $\text{Dim}_{\Delta}(X) < 1$, then X is Δ -meager.*
- (2) *If $\text{Dim}(X \mid R(\Delta)) < 1$, then X is meager in $R(\Delta)$.*

Proof. Part 2 is immediate from part 1. Assume the hypothesis of part 1. Then for some $s < 1$, there is an exactly- Δ -computable s -gale d with $X \subseteq S_{\text{str}}^{\infty}[d]$.

Let $t = \lceil \frac{s}{1-s} \rceil$. For each $w \in \{0, 1\}^*$, we inductively construct an extension w' of w by the following algorithm.

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 $w' := w.$ 
for  $i = 1$  to  $t|w|$ 
  if  $d(w'0) \leq d(w'1)$ 
     $w' := w'0.$ 
  else
     $w' := w'1.$ 

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Because d is an s -gale, the average of $d(w'0)$ and $d(w'1)$ is $2^{s-1}d(w')$. One of $d(w'0)$ and $d(w'1)$ must be no more than this average, so $d(w')$ decreases by multiplicative factor of 2^{s-1} (or a smaller factor) each iteration of the for-loop. Therefore, $d(w') \leq 2^{(s-1)t|w|}d(w)$ at the end.

We define a constructor $h : \{0, 1\}^* \rightarrow \{0, 1\}^*$ by

$$h(w) = w'.$$

Then $h \in \Delta$ by the above algorithm. For each $w \in \{0, 1\}^*$,

$$d(h(w)) \leq 2^{(s-1)t|w|} d(w) \leq 2^{-s|w|} d(w).$$

Also, $d(w) \leq 2^{s|w|} d(\lambda)$ because d is an s -gale, so we have

$$d(h(w)) \leq d(\lambda) \tag{3.1}$$

for every $w \in \{0, 1\}^*$.

Since $h \in \Delta$, it suffices to show that h always wins the Banach-Mazur game $G[X; \text{all}, \Delta]$ for player II. Let g be any constructor. Let $R(g, h)$ be the sequence built when g and h are played against each other. We need to show that $R(g, h) \notin X$. For this we can show $R(g, h) \notin S_{\text{str}}^\infty[d]$ since $X \subseteq S_{\text{str}}^\infty[d]$.

Define $w_0 = \lambda$ and $w_n = h(g(w_{n-1}))$ for all $n \geq 1$. Then each w_n is a prefix of $R(g, h)$ and for every $n \geq 1$,

$$d(w_n) = d(h(g(w_{n-1}))) \leq d(\lambda)$$

by (3.1). Therefore

$$\liminf_{n \rightarrow \infty} d(R(g, h) \upharpoonright n) \leq d(\lambda),$$

i.e., d does not succeed strongly on $R(g, h)$. \square

We remark that Theorem 3.1 does not extend to *resource-bounded genericity*, a different notion of resource-bounded category. For example, we might ask if strong $\text{Dim}_p(X) < 1$ implies that X has no p -generics. This is false because there are sparse n^2 -generics [11] but the class of sparse languages has strong p -dimension 0.

4 Corollaries

In general, Δ -measure 0 and Δ -meager are incomparable properties. For example, Mayordomo [12] showed that the class of non-P-bi-immune languages has p -measure 0 but is not p -meager. There are also examples of classes that are Δ -meager which do not have Δ -measure 0 [13]. However, Lutz [1,2] showed that several classes both have Δ -measure 0 and are Δ -meager. Proposition 2.5 and Theorem 3.1 give us the following corollary which along with recent work of Gu [9] provides further explanation of Lutz's results.

Corollary 4.1 *Let $X \subseteq \mathbf{C}$.*

- (1) If $\text{Dim}_\Delta(X) < 1$, then X has Δ -measure 0 and is Δ -meager.
(2) If $\text{Dim}(X \mid R(\Delta)) < 1$, then X has measure 0 in $R(\Delta)$ and is meager in $R(\Delta)$.

For example, Lutz showed that for each constant c , the circuit-size complexity class $\text{SIZE}(n^c)$ has p_2 -measure 0 and is p_2 -meager. Gu showed $\text{Dim}_{p_2}(\text{SIZE}(n^c)) = 0$. By Corollary 4.1, this yields a new proof of Lutz's simultaneous measure 0 and meager result. Similarly, Lutz showed that P/poly has p_3 -measure 0 and is p_3 -meager; Gu showed that $\text{Dim}_{p_3}(P/\text{poly}) = 0$.

However, we remark that the converse of Corollary 4.1 does not hold in general. For example, the class $\text{SIZE}(\frac{2^n}{n})$ has pspace-measure 0 and is pspace-meager [1], but it has pspace-dimension 1 [3].

For a class X of languages, define

$$\text{io-}X = \{A \subseteq \{0, 1\}^* \mid (\exists B \in X)(\exists^\infty n)A_{=n} = B_{=n}\}.$$

(Here $A_{=n} = A \cap \{0, 1\}^n$.) Gu [9] showed that if X contains the empty language, then $\text{dim}(\text{io-}X \mid R(\Delta)) \geq 1/2$ and $\text{Dim}(\text{io-}X \mid R(\Delta)) = 1$ for every Δ . Theorem 3.1 along with the resource-bounded Baire category theorem provides a simpler proof of (a minor extension of) the latter fact.

Corollary 4.2 (Gu [9]) *If $X \cap R(\Delta) \neq \emptyset$, then $\text{Dim}(\text{io-}X \mid R(\Delta)) = 1$.*

Proof. Let $B \in X \cap R(\Delta)$. Then

$$E = \{A \subseteq \{0, 1\}^* \mid (\exists^\infty n)A_{=n} = B_{=n}\}$$

is a subclass of $\text{io-}X$. Because $B \in R(\Delta)$, it can be shown that E is Δ -comeager. In particular, E^c is meager in $R(\Delta)$.

Suppose that $\text{Dim}(\text{io-}X \mid R(\Delta)) < 1$. Then $\text{io-}X$ is meager in $R(\Delta)$ by Theorem 3.1, so E is also meager in $R(\Delta)$. But since the meager sets are closed under union, the resource-bounded Baire category theorem tells us we cannot have both E and E^c meager in $R(\Delta)$. \square

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