Com S 633: Randomness in Computation	
Lecture 18	Scribe: Rakesh Setty

1 Spectral Expansion

In the last class, we mentioned a theorem which states that for every $n \times n$ real symmetric matrix M, there exists orthogonal vectors v_1 , v_2 , ... v_n that are eigen vectors of the matrix M. Let λ_1 , λ_2 , ... λ_n be the corresponding eigen values. Recall that these values may not be distinct.

From now, we use the following convention: We assume that all eigen vectors are ordered according to their absolute values. That is we have $|\lambda_1| \geq |\lambda_2| \geq \cdots \geq |\lambda_n|$.

For every $v \in R^n$

$$v = a_1 v_1 + a_2 v_2 + \dots a_n v_n$$

Now,

$$\begin{aligned} Mv &= a_1 M v_1 + a_2 M v_2 + \dots a_n M v_n \\ &= a_1 \lambda_1 v_1 + a_2 \lambda_2 v_2 + \dots a_n \lambda_n v_n \\ &= \lambda_1 a_1 v_1 + \lambda_2 a_2 v_2 + \dots \lambda_n a_n v_n \end{aligned}$$

That is if we represent v with respect to eigen basis of M, then M is just stretching the vector v along each co-ordinate.

Observation: $\forall v$, $||Mv||_2 \leq |\lambda_1| \, ||v||_2$

Now, look at the eigen space of λ_1. Consider a vector v_P that is perpendicular to this sub space. Thus v_P can be expressed as

$$v_P = a_2 v_2 + a_3 v_3 + \dots a_n v_n$$

Now

$$||M v_P||_2 \leq |\lambda_2| \, ||v||_2$$

Thus every vector that is orthogonal to v_1 is stretched by at most $|\lambda_2$ by M. This observation turns out to be very crucial.

1.1 Eigen values and Graphs

Let G be a d-regular undirected multi-graph. Multigraph is a graph in which more than one edge could exist between a pair of vertices. We define a normalized adjacency matrix M_G of G as follows.

$$M_G\,(i\,,\,j) = \frac{n_{ij}}{d}$$

where $M_G\,(i\,,\,j)$ represents the element $(i\,,\,j)$ of the matrix M_G and n_{ij} represents the number of edges from the vertex i to vertex j in the graph G.

Observation: We can observe that

$$M^G \left[\frac{1}{n}\,,\,\frac{1}{n}\,,\,\dots\,\frac{1}{n}\right]^T \;=\; \left[\frac{1}{n}\,,\,\frac{1}{n}\,,\,\dots\,\frac{1}{n}\right]^T$$

This shows that 1 is an eigenvalue of M_G and the vector $\left\langle \frac{1}{n}\,,\,\frac{1}{n}\,,\,\dots\,\frac{1}{n} \right\rangle$ is a corresponding eigenvector.

Theorem 1.1. *If G is a d-regular undirected multigraph, then*

1. $\forall i\,,\, |\lambda_i| \;\leq\; 1$.
2. *G is connected iff multiplicity of λ_1 is* 1.
3. *G is bipartite iff -1 is an eigen value of M_G.*

Proof. (1) Let λ be any eigenvalue of M_G and $v \;=\; \langle v_1\,,\, v_2\,,\,\dots\, v_n\rangle$ be the corresponding eigenvector. Then we have

$$M_G v = \lambda v$$

Let $v_m \in \{v_1\,,\, v_2\,,\,\dots\, v_n\}$ such that

$$|v_m| \;=\; \max_i\{|v_i|\}$$

Then,

$$\begin{aligned} |\lambda v_m| &= \left|\sum_{i=1}^{n} a_{mi} v_i\right| \\ |\lambda|\,|v_m| &\leq \sum_{i=1}^{n} |a_{mi}|\,|v_i| \\ &\leq \left(\sum_{i=1}^{n} |a_{mi}|\right) |v_m| \\ |\lambda|\,|v_m| &\leq |v_m| \end{aligned}$$

Since $|v_m| \neq 0$, we get

$$|\lambda| \leq 1.$$

Since, we know that 1 is an eigenvalue, we have $\lambda_1 \;=\; 1$

(2) ***if* part**

Suppose G is disconnected. Let it consist of two components G_1 and G_2 with m and l vertices respectively with $\{v_1\,,\, v_2\,,\,\dots\, v_m\}$ and $\{v_{m+1}\,,\, v_{m+2}\,,\,\dots\, v_n\}$.
The matrix M_G will look like

$$\begin{bmatrix} \frac{1}{d_1} & 0 & \dots & \frac{1}{d_1} & 0 & 0 & \dots & 0 \\ \vdots \\ \vdots \\ 0 & 0 & \dots & 0 & \frac{1}{d_2} & 0 & \dots & \frac{1}{d_2} \\ \vdots \\ \vdots \end{bmatrix}$$

We can see that

$$\begin{bmatrix} \frac{1}{d_1} & 0 & \dots & \frac{1}{d_1} & 0 & 0 & \dots & 0 \\ \vdots \\ \vdots \\ 0 & 0 & \dots & 0 & \frac{1}{d_2} & 0 & \dots & \frac{1}{d_2} \\ \vdots \\ \vdots \end{bmatrix} \cdot \begin{bmatrix} \frac{1}{m} \\ \vdots \\ \frac{1}{m} \\ 0 \\ \vdots \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{1}{m} \\ \vdots \\ \frac{1}{m} \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

and

$$\begin{bmatrix} \frac{1}{d_1} & 0 & \dots & \frac{1}{d_1} & 0 & 0 & \dots & 0 \\ \vdots \\ \vdots \\ 0 & 0 & \dots & 0 & \frac{1}{d_2} & 0 & \dots & \frac{1}{d_2} \\ \vdots \\ \vdots \end{bmatrix} \cdot \begin{bmatrix} 0 \\ \vdots \\ 0 \\ \frac{1}{l} \\ \vdots \\ \frac{1}{l} \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ \frac{1}{l} \\ \vdots \\ \frac{1}{l} \end{bmatrix}$$

Hence, both $\left\langle \frac{1}{m}, \frac{1}{m}, \dots \frac{1}{m}, 0, \dots 0 \right\rangle$ and $\left\langle 0, \dots 0, \frac{1}{l}, \frac{1}{l}, \dots \frac{1}{l} \right\rangle$ are eigen vectors with same eigenvalue 1. Since they are orthogonal to each other, the multiplicity of λ_1 is at least 2.

***Only if* part** Suppose G is connected and the multiplicity of λ_1 is more than 1, i.e. there is at least one more vector $\langle x_1, x_2, \dots x_n \rangle$ that is orthogonal to the vector $\left\langle \frac{1}{n}, \frac{1}{n}, \dots \frac{1}{n} \right\rangle$ and whose eigen value is 1.
i.e.

$$\begin{bmatrix} \frac{1}{d} & 0 & \dots & \frac{1}{d} & \dots \\ \vdots \\ \vdots \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

It can not be the case that all x_i's are equal, otherwise this vector is not orthogonal to $\langle 1, 1, \cdots 1\rangle$. Let x_m be the largest among $\{x_1\,,\,x_2\,,\,\ldots\,x_n\}$. We show that if x_i is less than x_m, then there is no edge from v_i to v_m. Observe that

$$\sum M_{mj} x_j = x_m$$

Observe that each m_{mj} is either 0 or $1/d$, and each $x_j \leq x_m$, and there is at least one x_i for which $x_i < x_j$. Since there are exactly d M_{mj}'s that are not equal to zero, the above equation is satisfied only if $M_{mi} = 0$. Thus there is no edge from v_m to v_i. If $W = v_i \mid x_i = x + m$, then no edge goes out of W. Thus G is not connected.

(3) ***Only if* part**

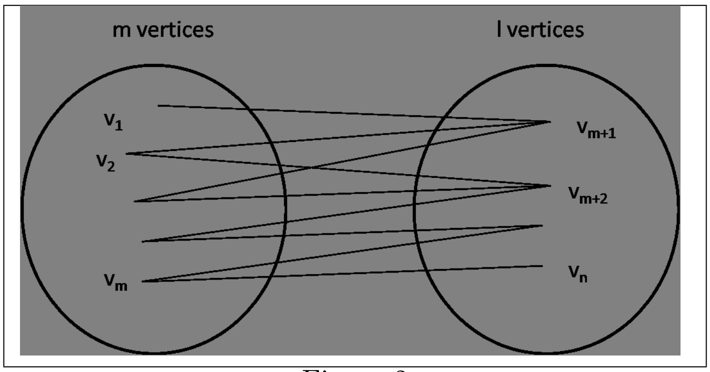

Figure 2

Suppose $G = (L, R, E)$ is bipartite then define $v = (v_1, v_2, \ldots, v_n)$ where

$$v_i = \begin{cases} 1 & \text{if } i \in L \\ -1 & \text{if } i \in R \end{cases}$$

Now if $v_i \in L$, then $(Av)_i = \sum_{j=1}^{n} a_{ij} v_j = \sum_{\substack{j=1 \\ j \in R}}^{n} a_{ij} v_j = -1$. Similarly, if $v_i \in R$ then $(Av)_i = 1$. Thus, $Av = -v$ and -1 is eigenvalue of A.

***if* part**

Suppose that -1 is eigenvalue of A and $x = \langle x_1, \cdots, x_n\rangle$ be the corresponding eigenvector, then we have $Av = -v$.

Let $x_m \in \{x_1\,,\,x_2\,,\,\ldots\,x_n\}$ such that

$$||x_m|| \;=\; \max_i ||x_i||$$

By using argument similar to the one above, we can show that if $(m, i) \in E$ then of $x_i = -x_m$. Now if $(m, i) \in E$ and $(m, j) inE$, then $x_i = x_j = -x_m$. If there is an edge from i to j, then $x_i = -x_j$, which is a contradiction. This implies that G is bi-partite. □

1.2 Spectral Gap and its significance

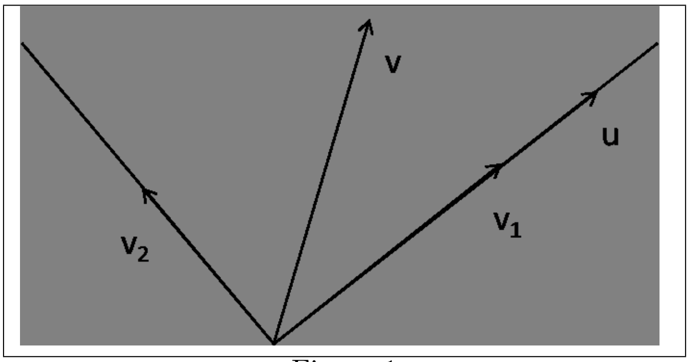

Figure 1

Lets say we have a vector u and another vector v. We want to know how "close" is v to u. Consider two components of v, v_1 and v_2 where v_1 is along u and v_2 is perpendicular to u. We know that

$v = v_1 + v_2$

The smaller the v_2, closer the v is to u. Thus lower the L_2 norm of v_2, the closer v is to u.

Say is u is the vector $\langle 1/\sqrt{n}, 1/\sqrt{n}, \cdots 1/\sqrt{n} \rangle$. And say M is a real symmetric matrix. What happens when we apply M to v? More precisely, we want to know how close if Mv to u. As above $v = v_1 + v_2$, where v_1 is parallel to u and v_2 is perpendicular to u. Note that the eigen value of u is 1. Thus the vector v_2 is perpendicular to the eigen space of 1. Now $MV = Mv_1 + Mv_2$. Observe that $Mv_1 = v_1 1/\sqrt{n}$. Since v_2 is perpendicular to u, $||Mv_2||_2 \leq |\lambda_2| ||v_2||_2$. If $|\lambda_2| < 1$, then MV is "more close" to u than v was from u.

we make these ideas more explicit below.

Let G be a connected d-regular non-bipartite graph. Let M_G be the corresponding normalized adjacency matrix. So, $\lambda_1 = 1$ and $|\lambda_2| < 1$. Observe that $|\lambda_1| > |\lambda_2| \geq |\lambda_3| \geq \ldots |\lambda_n|$. Let $v_1, \cdots, v_n$ be orthonormal eigen vectors corresponding to eigen values $\lambda_1, \cdots \lambda_n$. We will take v_1 as $\langle 1/\sqrt{n}, \cdots 1/\sqrt{n} \rangle$.

Definition:

Spectral Gap of a graph G is $|\lambda_2|$ denoted by $\lambda(G)$.

Let P be a probability distribution over $\{1, 2, \ldots n\}$. p_i indicates the probability of element i. We can think of P as a vector $\langle p_1, p_2, \ldots p_n \rangle$.

If we take a random walk on G, then the probability distribution after one step is MP.

Let $Unif = \left\langle \frac{1}{n}, \frac{1}{n}, \ldots \frac{1}{n} \right\rangle$ be the uniform distribution vector.

Let $P = a_1 v_1 + a_2 v_2 + \ldots a_n v_n$

Then,

$$MP = a_1 M v_1 + a_2 M v_2 + \ldots a_n M v_n$$

We know that eigen value of v_1 is 1. Thus $Mv_1 = v_1$. Now

$$MP = a_1 v_1 + a_2 M v_2 + \ldots a_n M v_n$$

$MP - a_1 v_1 = a_2 M v_2 + \ldots a_n M v_n$

$$
\begin{aligned}
||MP - a_1v_1||_2 &= \sqrt{\sum_{i=2}^{n} \left(\lambda_i {a_i}^2\right)} \\
&\leq \lambda \sqrt{\sum_{i=2}^{n} {a_i}^2} \\
&= \lambda\, ||P - a_1v_1||_2 \\
&\leq \lambda\, ||P||_2 \\
&\leq \lambda\, ||P||_1 \\
&= \lambda
\end{aligned}
$$

Thus

$$||MP - a_1v_1||_2 \leq \lambda$$

We know that,

$$
\begin{aligned}
a_1 &= \langle Pv_1 \rangle \\
&= \langle p_1\,,\, p_2\,,\, \dots\, p_n \rangle \cdot \left\langle \frac{1}{\sqrt{n}}\,,\, \frac{1}{\sqrt{n}}\, \cdots\, \frac{1}{\sqrt{n}} \right\rangle \\
&= \frac{\sum_{i=1}^{n} p_i}{\sqrt{n}} \\
&= \frac{1}{\sqrt{n}}
\end{aligned}
$$

Thus,

$$a_1v_1 = \left\langle \frac{1}{n}\,,\, \frac{1}{n}\,,\, \dots\, \frac{1}{n} \right\rangle$$

Therefore

$$||MP - Unif||_2 = ||MP - a_1v_1||_2$$

We know that,

$$
\begin{aligned}
||MP - Unif||_\infty &\leq ||MP - Unif||_2 \\
&\leq \lambda
\end{aligned}
$$

If we start with any probability distribution P and take one-step random walk, then the L_∞ norm of $MP - Unif$ at most λ.

Now observe that

$$\left|\left|MP^l - Unif\right|\right|_\infty \leq \left|\left|MP^l - Unif\right|\right|_2 \leq \lambda^l$$

Thus after a l-step random walk, each vertex is visited with probability at least $1/n - \lambda^l$ and at most with probability $1/n + \lambda^l$.

If we take $\lambda^l \simeq \frac{1}{2n}$,

$l = \frac{log\, n}{1 - \lambda}$ (with some approximation)

i.e. we will approach uniform distribution within $\frac{log\, n}{1 - \lambda}$ steps.

We will also mention the following theorem without proving.

Theorem 1.2. *For every d-regular connected non-bipartite graph,*

$$1 - \lambda \simeq \frac{1}{dn^2}$$

Thus if G is an undirected, connected, d-regular, non-bipartite graph then a $O(dn^2 \log n)$-step random walk yields a distribution on vertices of G, and this distribution is close to uniform distribution.