Com S 633: Randomness in Computation	
Lecture 17	Scribe: Ankit Agrawal

1 Vector Spaces

Definition: A vector over $\Re^n$ is a n-tuple $V =< x_1, x_2, ..., x_n >$ where each $x_i \in \Re$. Hence, a vector can be understood to be a line from the origin to a n-dimensional point in the n-dimensional space.

1.1 Operations on Vectors

Multiplication by a real number:
Let $a \in \Re$, $V \in \Re^n$, $V =< x_1, x_2, ..., x_n >$, then

$$aV =< ax_1, ax_2, ..., ax_n >$$

Addition of two vectors:
Let $V =< x_1, x_2, ..., x_n >$, $U =< y_1, y_2, ..., y_n >$, then

$$V + U =< x_1 + y_1, x_2 + y_2, ..., x_n + y_n >$$

Norms of a vector:
Let $V =< x_1, x_2, ..., x_n >$, then

$$\begin{aligned}
\|V\|_1 &= L_1\, norm\, of\, V = \sum_{i=1}^{n} |x_i| \\
\|V\|_2 &= L_2\, norm\, of\, V = \sqrt{\sum_{i=1}^{n} |x_i^2|} \\
&\vdots \\
\|V\|_\infty &= L_\infty\, norm\, of\, V = max_{i \in 1,...,n} |x_i|
\end{aligned}$$

$\|V\|_2$ also denotes the *length* of V.

Dot product: Given two vectors $U =< x_1, x_2, ..., x_n >$, and $V =< y_1, y_2, ..., y_n >$, the dot product of U and V is given by

$$U.V = \langle U, V \rangle = \sum_{i=1}^{n} x_i y_i$$

The dot product of two vectors is a number, and not a vector.
Dot product is *distributive*, i.e., given three vectors U, V, W, and a scalar a,

$$\langle aU + V, W \rangle = a \langle U, W \rangle + \langle V, W \rangle$$

If θ is the angle between two vectors U and V, then

$$\langle U, V \rangle = \|U\|_2 \|V\|_2 cos\theta$$

For every vector V, $L_\infty(V) \leq L_2(V) \leq L_1(V)$ (sum of squares is less than square of sums)

Cauchy-Schwartz Inequality: $L_1(V) \leq \sqrt{n} L_2(V)$
Proof: Let $V = < x_1, x_2, ..., x_n >$, $U = < y_1, y_2, ..., y_n >$. Let y_i equals 1 if $x_i \geq 0$, and -1 otherwise. Then,

$$\begin{aligned}
L_2(V) L_2(U) cos\theta &= \langle U, V \rangle = \sum_{i=1}^{n} |x_i| = L_1(V) \\
L_2(U) &= \sqrt{n} \\
\sqrt{n} L_2(V) cos\theta &= L_1(V) \\
0 \leq & \; cos\theta \leq 1 \\
\Rightarrow L_1(V) &\leq \sqrt{n} L_2(V)
\end{aligned}$$

Definition: A collection of vectors $v_1, v_2, ... v_i$ are *linearly dependent* if $\exists$ scalars $c_1, c_2, ... c_i$ not all zero s.t. $c_1 v_1 + c_2 v_2 + ... c_i = \overline{0}$ where $\overline{0}$ is the zero vector ($< 0, 0, ..., 0 >$).

Definition: A collection of vectors is *linearly independent* if they are not linearly dependent.

Definition: Given $V_1, V_2, ..., V_n$ (may or may not be independent). The *span* of these n vectors is defined as

$$span(V_1, V_2, ..., V_n) = \{c_1 V_1 + c_2 V_2 + \cdots + c_n V_n | c_1, c_2, ..., c_n \in \Re\}$$

i.e., the span of a set of vectors is the set of all possible linear combinations of those vectors.

Definition: A collection of vectors is a *subspace* if the collection is closed under scalar multiplication and addition. e.g. set of all vectors in $\Re^3$ with z-coordinate 0.

Definition: Let S be a subspace of $\Re^n$ and $v_1, v_2, ..., v_k$ be a set of vectors. We say $v_1, v_2, ..., v_k$ form a *basis* for S if $span(v_1, v_2, ..., v_k) = S$. e.g. for the subspace of the set of all vectors in $\Re^3$ with z-coordinate 0, a basis can be $< 1, 0, 0 >$, and $< 0, 1, 0 >$.

Definition: The *dimension* of a subspace S is the cardinality of the smallest basis for S.

Corollary: Every vector in subspace S with cardinality m can be written as a linear combination of the m basis vectors.

Observation: Dimension of $\Re^n = n$

Observation: There can be more than one smallest basis. e.g. for $\Re^2 : (< 1, 0 >, < 0, 1 >)$ and $(< 1, 0 >, < 1, 2 >)$.

Observation: The entire subspace is a basis for itself.

Definition: The Standard *Kronecker Basis* for $\Re^n$ is

$$
\begin{array}{c}
< 1, 0, 0, ..., 0 > \\
< 0, 1, 0, ..., 0 > \\
< 0, 0, 1, ..., 0 > \\
\vdots \\
< 0, 0, 0, ..., 1 >
\end{array}
$$

This is based on the Kronecker delta functions:
$\delta_i(j) = 1$ if $i = j$, and 0 otherwise

Definition: Let S be a subspace, and let $V = \{v_1, v_2, ..., v_k\}$ be a basis for S. We say V is an *orthogonal basis* if $\forall i \neq j \ \langle v_i, v_j \rangle = 0$, i.e., v_i and v_j are perpendicular.

Observation: The Kronecker basis is orthogonal.

Definition: A basis is *orthonormal* if it is an orthogonal basis and all basis vectors are of unit length.

Matrix representation of a vector: A vector $V =< x_1, x_2, ..., x_n >$ can be represented in the matrix format as a column vector:

$$
\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}
$$

Observation: $\langle U, V \rangle = U^T V$

Observation: $\|V\|_2 = \sqrt{\langle V, V \rangle}$

Let V_1, V_2, V_3 be a basis for $\Re^3$, and let V be a vector. Then, $V = a_1 V_1 + a_2 V_2 + a_3 V_3$. Now we can define Norm of vector V with respect to this new basis:

$$
\begin{array}{r}
L_1 \text{ norm of } V \text{ w.r.t. basis } V_1, V_2, V_3 = \sum_{i=1}^{3} |a_i| \\
L_2 \text{ norm of } V \text{ w.r.t. basis } V_1, V_2, V_3 = \sqrt{\sum_{i=1}^{3} a_i^2}
\end{array}
$$

This notion can be generalized to n-dimensional space. An obvious question is whether a norm of a vector changes as we change the basis. In general the answer is "Yes". However, in certain scenarios the norm of a vector (L_2 norm) remains unchanged.

Lemma: Say $V_1, V_2, ...V_n$ be an orthonormal basis for $\Re^n$. Let $V = \sum_{i=1}^{n} a_i V_i$. We have $L_2(V)$ w.r.t. $V_1, V_2, ...V_n = \sqrt{\sum_{i=1}^{n} a_i^2}$. Then,

$$
L_2(V) \text{ w.r.t. } V_1, V_2, ..., V_n = L_2(V) \text{ w.r.t. the standard basis.}
$$

i.e., L_2 norm is invariant w.r.t. orthonormal basis.

To prove this we first need the following claim.

Claim: Let $V_1, V_2, ...V_n$ be an orthonormal basis. $\forall\, U \,\exists\, a_1, a_2, ..., a_n$ s.t. $U = a_1 V_1 + a_2 V_2 +$

$\cdots + a_n V_n$. Then, $a_i = \langle U, V_i \rangle$.
Proof:

$$\begin{aligned}
\langle U, V_i \rangle &= \langle a_1 V_1 + a_2 V_2 + \cdots + a_n V_n, V_i \rangle \\
&= a_1 \langle V_1, V_i \rangle + a_2 \langle V_2, V_i \rangle + \cdots + a_i \langle V_i, V_i \rangle + \cdots + a_n \langle V_n, V_i \rangle \\
&= 0 + 0 + \cdots + a_i \|V_i\|_2^2 + \cdots + 0 \; [since\, V_1, V_2, ...V_n\, is\, orthonormal\, basis] \\
&= a_i
\end{aligned}$$

Now we can show that L_2 norm remains the same with respect to all orthonormal bases. Let V be any vector and let $V_1, \cdots, v_n$ be orthonormal basis. By previous claim, $V = a_1 V_1 + \cdots + a_n V_n$, where $a_i = \langle V V_i \rangle$. Recall that $\langle V, V_i \rangle = V^T V_i$. Thus L_2 norm of V wrt the basis $V_1, \cdots, V_n$ is

$$\begin{aligned}
\sum [V^T V_i]^2 &= V^T V_1 V^T V_1 + \cdots + V^T V_n V^T V_n \\
&= V^T [V_1 V^T V_1 + \cdots V_n V^T V_n] \\
&= V^T [V_1 \langle V, V_1 \rangle + \cdots V_n \langle V, V_n \rangle] \\
&= V^T [a_1 V_1 + \cdots + a_n V_n] \\
&= V^T V \\
&= ||V||_2
\end{aligned}$$

Definition: Let A be a $n \times n$ matrix. λ is an *eigen value* of A if there is a non-zero vector V s.t.

$$AV = \lambda V$$

V is called the eigen vector associated with λ.

Definition: Let λ be an eigen value. Let S_λ be the collection of all eigen-vectors associated with λ. S_λ is called *eigen-space* of λ.

Observation: S_λ is a subspace.

Definition: *Multiplicity* of eigen-value λ is the dimension of subspace S_λ.

Theorem: Let M be a real, symmetric matrix. All its eigen values are real.

It is possible to have complex eigen values for real matrices. This is because the eigen values are the roots of a polynomial given by the determinant of the matrix $A - \lambda I$, where I is the identity matrix. And in general, the polynomial may have complex roots.

Claim: Let M be a real symmetric matrix, and $\lambda_1 \neq \lambda_2$ be two distinct eigen values. Let V_1 and V_2 be two eigen vectors corresponding to λ_1 and λ_2. Then,

$$\langle V_1, V_2 \rangle = 0$$

Proof:

$$\begin{aligned}
\lambda_1 \langle V_1, V_2 \rangle &= \langle \lambda_1 V_1, V_2 \rangle \\
&= \langle M V_1, V_2 \rangle \\
&= (M V_1)^T V_2 \\
&= V_1^T M^T V_2 \quad [since\, (AB)^T = B^T A^T] \\
&= V_1^T M V_2 \qquad [since\, M\, is\, symmetric] \\
&= \langle V_1, M V_2 \rangle \\
&= \langle V_1, \lambda_2 V_2 \rangle \\
&= \lambda_2 \langle V_1, V_2 \rangle \\
\Rightarrow \lambda_1 \langle V_1, V_2 \rangle &= \lambda_2 \langle V_1, V_2 \rangle \\
\Rightarrow \langle V_1, V_2 \rangle &= 0 \qquad\qquad [since\, \lambda_1 \neq \lambda_2]
\end{aligned}$$

Theorem: Let M be a $n \times n$ real-symmetric matrix. Then, there exist orthonormal vectors $V_1, V_2, ..., V_n$ such that each V_i is an eigen-vector of M.