Solution to Assignment 8

Com S 477/577

Nov 5, 2010

1 . We apply Gram-Schmidt procedure to find orthonormal vectors $\boldsymbol{u}_1$, $\boldsymbol{u}_2$ and $\boldsymbol{u}_3$.

$$\boldsymbol{w}_1 = \boldsymbol{v}_1 \qquad \boldsymbol{u}_1 = \frac{\boldsymbol{w}_1}{|\boldsymbol{w}_1|} = \left(\frac{2}{3}, \frac{2}{3}, \frac{1}{3}\right)^T,$$

$$\begin{aligned}\boldsymbol{w}_2 &= \boldsymbol{v}_2 - \frac{\boldsymbol{v}_2^T \boldsymbol{w}_1}{\boldsymbol{w}_1^T \boldsymbol{w}_1}\boldsymbol{w}_1 \\ &= \left(-\frac{5}{3}, \frac{1}{3}, \frac{8}{3}\right)^T \qquad \boldsymbol{u}_2 = \frac{\boldsymbol{w}_2}{|\boldsymbol{w}_2|} = \left(-\frac{5}{3\sqrt{10}}, \frac{1}{3\sqrt{10}}, \frac{8}{3\sqrt{10}}\right)^T,\end{aligned}$$

$$\begin{aligned}\boldsymbol{w}_3 &= \boldsymbol{v}_3 - \frac{\boldsymbol{v}_3^T \boldsymbol{w}_2}{\boldsymbol{w}_2^T \boldsymbol{w}_2}\boldsymbol{w}_2 - \frac{\boldsymbol{v}_3^T \boldsymbol{w}_1}{\boldsymbol{w}_1^T \boldsymbol{w}_1}\boldsymbol{w}_1 \\ &= \left(\frac{1}{9}, -\frac{7}{45}, \frac{4}{45}\right)^T \qquad \boldsymbol{u}_3 = \frac{\boldsymbol{w}_3}{|\boldsymbol{w}_3|} = \left(\frac{\sqrt{5}}{3\sqrt{2}}, -\frac{7\sqrt{5}}{15\sqrt{2}}, \frac{4\sqrt{5}}{15\sqrt{2}}\right)^T.\end{aligned}$$

Thus we have

$$\begin{aligned}
\boldsymbol{v}_1 &= \boldsymbol{w}_1 \\
&= 3\boldsymbol{u}_1, \\
\boldsymbol{v}_2 &= \frac{\boldsymbol{v}_2^T \boldsymbol{w}_1}{\boldsymbol{w}_1^T \boldsymbol{w}_1}\boldsymbol{w}_1 + \boldsymbol{w}_2 \\
&= \frac{12}{9} \cdot 3\boldsymbol{u}_1 + \sqrt{10}\boldsymbol{u}_2 \\
&= 4\boldsymbol{u}_1 + \sqrt{10}\boldsymbol{u}_2, \\
\boldsymbol{v}_3 &= \frac{\boldsymbol{v}_3^T \boldsymbol{w}_1}{\boldsymbol{w}_1^T \boldsymbol{w}_1}\boldsymbol{w}_1 + \frac{\boldsymbol{v}_3^T \boldsymbol{w}_2}{\boldsymbol{w}_2^T \boldsymbol{w}_2}\boldsymbol{w}_2 + \boldsymbol{w}_3 \\
&= \frac{5}{9}\boldsymbol{w}_1 + \frac{2}{15}\boldsymbol{w}_2 + \boldsymbol{w}_3 \\
&= \frac{5}{3}\boldsymbol{u}_1 + \frac{2\sqrt{10}}{15}\boldsymbol{u}_2 + \frac{\sqrt{10}}{15}\boldsymbol{w}_3.
\end{aligned}$$

The QR decomposition for the matrix is then constructed as

$$\begin{aligned}
\begin{pmatrix} 2 & 1 & 1 \\ 2 & 3 & 1 \\ 1 & 4 & 1 \end{pmatrix} = QR &= (\boldsymbol{u}_1\ \boldsymbol{u}_2\ \boldsymbol{u}_3) \begin{pmatrix} 3 & 4 & \frac{5}{3} \\ 0 & \sqrt{10} & \frac{2\sqrt{10}}{15} \\ 0 & 0 & \frac{\sqrt{10}}{15} \end{pmatrix} \\
&= \frac{1}{3\sqrt{10}} \begin{pmatrix} 2\sqrt{10} & -5 & 5 \\ 2\sqrt{10} & 1 & -7 \\ \sqrt{10} & 8 & 4 \end{pmatrix} \cdot \begin{pmatrix} 3 & 4 & \frac{5}{3} \\ 0 & \sqrt{10} & \frac{2\sqrt{10}}{15} \\ 0 & 0 & \frac{\sqrt{10}}{15} \end{pmatrix}.
\end{aligned}$$

2. (a) Please see the Assignments page for an example.

   Note that the pivoting step exchanges rows whenever necessary, otherwise the decomposition of a nonsingular matrix could be unsolvable.

$$PA = LU,$$

   where

$$P = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

$$L = \begin{pmatrix} 1.00000 & 0.00000 & 0.00000 & 0.00000 & 0.00000 \\ 0.26829 & 1.00000 & 0.00000 & 0.00000 & 0.00000 \\ -0.07317 & -0.46193 & 1.00000 & 0.00000 & 0.00000 \\ 0.02439 & 0.09848 & 0.32086 & 1.00000 & 0.00000 \\ -0.14634 & -0.96548 & -0.09613 & -0.27799 & 1.00000 \end{pmatrix},$$

$$U = \begin{pmatrix} 41.00000 & 97.00000 & -32.00000 & 47.00000 & 23.00000 \\ 0.00000 & -24.02439 & 3.58537 & -6.60976 & 41.82927 \\ 0.00000 & 0.00000 & 54.31472 & -60.61421 & 21.00508 \\ 0.00000 & 0.00000 & 0.00000 & 47.95323 & -32.41989 \\ 0.00000 & 0.00000 & 0.00000 & 0.00000 & 86.75815 \end{pmatrix}.$$

   (b)

$$(1)\ \boldsymbol{x} = \begin{pmatrix} 0.75298 \\ -0.33646 \\ -0.09446 \\ -0.02951 \\ -0.08136 \end{pmatrix} \quad \text{and} \quad (2)\ \boldsymbol{x} = \begin{pmatrix} -9.34323 \\ -3.31220 \\ -9.06090 \\ 8.51527 \\ 0.31258 \end{pmatrix}$$

3. The real root of p is -0.04531644762684 (other roots are $0.12720367835887 \pm 0.88671142587521i$).

   The real roots of q are -3.003537406197627, 2.987813177502642 (other roots are $-0.0077192810013444 \pm 2.96831864814176i$).

4. (a) Please see Assignments page for examples.

   (b) The roots of p are 1.7, $1+i$, $1-i$, $1.414214i$ and $-1.414214i$, and the roots of q are ± 1.414214, ± 10, ± 8, and ± 2.

5. Part 1. A collision happens when the center distance between the cue ball $(x(t), y(t))$ and the object ball $(x_{\text{obj}}, y_{\text{obj}})$ is exactly $2R$. Thus the real roots of the collision equation

$$(x(t) - x_{\text{obj}})^2 + (y(t) - y_{\text{obj}})^2 - 4R^2 = 0$$

   represent the time of the collision. Based on the trajectory functions, we know that the collision equation is a quartic equation which has roots in closed form.

   Since there are two phases to describe the trajectory of the cue ball, we first check if there are real roots of the collision equation that are between starting time (0) and t_1. We only need the smallest valid collision time.

   If the collision is found during the sliding phase, we can stop; otherwise, we repeat the same process on the rolling phase, except that the collision time (real root) is only valid between t_1 and t_2.

   If there is no valid root found in the above two phases, then there is no collision.

The sliding collision equation has the form, $0 \le t \le t_1$,

$$
\begin{aligned}
&-4R^2 + (x_0 - x_{\text{obj}})^2 + (y_0 - y_{\text{obj}})^2 \\
&+ 2\boldsymbol{v}_{0,x}(x_0 - x_{\text{obj}})t + 2\boldsymbol{v}_{0,y}(y_0 - y_{\text{obj}})t \\
&- g\mu_s\hat{\boldsymbol{u}}_{0,x}(x_0 - x_{\text{obj}})t^2 - g\mu_s\hat{\boldsymbol{u}}_{0,y}(y_0 - y_{\text{obj}})t^2 + (\boldsymbol{v}_{0,y}^2 + \boldsymbol{v}_{0,x}^2)t^2 \\
&- g\mu_s(\hat{\boldsymbol{u}}_{0,y}\boldsymbol{v}_{0,y} + \hat{\boldsymbol{u}}_{0,x}\boldsymbol{v}_{0,x})t^3 + \frac{1}{4}g^2{\mu_s}^2(\hat{\boldsymbol{u}}_{0,y}^2 + \hat{\boldsymbol{u}}_{0,x}^2)t^4 = 0 \ .
\end{aligned}
$$

The rolling collision equation has the form, $t_1 \le t \le t_2$,

$$
\begin{aligned}
&-4R^2 + (x_1 - x_{\text{obj}})^2 + (y_1 - y_{\text{obj}})^2 \\
&+ 2\boldsymbol{v}_{1,x}(x_1 - x_{\text{obj}})(t - t_1) + 2\boldsymbol{v}_{1,y}(y_1 - y_{\text{obj}})(t - t_1) \\
&- \frac{5}{7}g\mu_r\hat{\boldsymbol{v}}_{1,x}(x_1 - x_{\text{obj}})(t - t_1)^2 - \frac{5}{7}g\mu_r\hat{\boldsymbol{v}}_{1,y}(y_1 - y_{\text{obj}})(t - t_1)^2 + (\boldsymbol{v}_{1,x}^2 + \boldsymbol{v}_{1,y}^2)(t - t_1)^2 \\
&- \frac{5}{7}g\mu_r(\hat{\boldsymbol{v}}_{1,x}\boldsymbol{v}_{1,x} + \hat{\boldsymbol{v}}_{1,y}\boldsymbol{v}_{1,y})(t - t_1)^3 + \frac{25}{196}g^2\mu_r^2(\hat{\boldsymbol{v}}_{1,x}^2 + \hat{\boldsymbol{v}}_{1,y}^2)(t - t_1)^4 = 0.
\end{aligned}
$$

We can also determine the angular velocity once the collision time determined.

During the rolling phase, we know there is no contact velocity, hence the angular velocity can be expressed as $\boldsymbol{w}(t) = (-\frac{v_y(t)}{R}, \frac{v_x(t)}{R})$, where $(v_x(t), v_y(t))$ can be determined by the first derivative of the rolling trajectory.

During the sliding phase, the sliding friction is a constant force applied to the cue ball in a constant direction, until the contact velocity becomes zero. Thus the angular velocity declines linearly over time from $\boldsymbol{w}_0$ to $\boldsymbol{w}(t_1)$.

Part 2.

(a)

Sliding trajectory: $(x(t), y(t)) = (-0.15635t^2 + 2t, 0.71869t^2)$

Rolling trajectory: $(x(t), y(t)) = (-0.06270t^2 + 1.88287t + 0.03662, -0.03123t^2 + 0.93792t - 0.29326)$

Other key values related to the trajectory:

$$
\begin{aligned}
\boldsymbol{u}_0 &= (0.68440, -3.14600) & \hat{\boldsymbol{u}}_0 &= (0.21257, -0.97714) \\
\boldsymbol{v}_1 &= (1.80446, 0.89886) & \hat{\boldsymbol{v}}_1 &= (0.89510, 0.44588) \\
(x_1, y_1) &= (1.18955, 0.28105) & (x_2, y_2) &= (14.17244, 6.74823) \\
t_1 &= 0.625345090107 & t_2 &= 15.0151443235
\end{aligned}
$$

There are collisions in (b) at $t = 0.48$ where

$$\boldsymbol{x} = (0.92459, 0.16581), \qquad \boldsymbol{v} = (1.8498, 0.69042), \qquad \boldsymbol{\omega} = (-49.649, 59.129),$$

and (d) at $t = 4.80$ where

$$\boldsymbol{x} = (7.62725, 3.48787), \qquad \boldsymbol{v} = (1.28121, 0.63821), \qquad \boldsymbol{\omega} = (-22.3152, 44.7977).$$

No collision in (c).

The trajectory of the cue ball and the position of the object balls is illustrated in Figure 1.

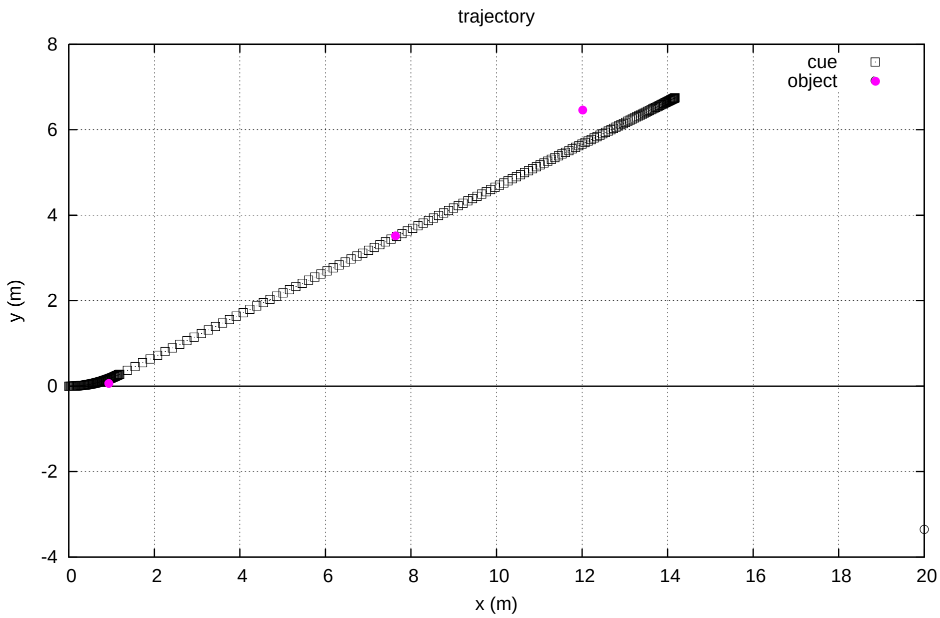

Figure 1: (courtesy of Wen-Chieh Chang) The trajectory of the cue ball and the positions of three object balls in Problem 5. Note that the point size in the chart does not reflect the size of balls.