# Solution to Assignment 7

**Com S 477/577**

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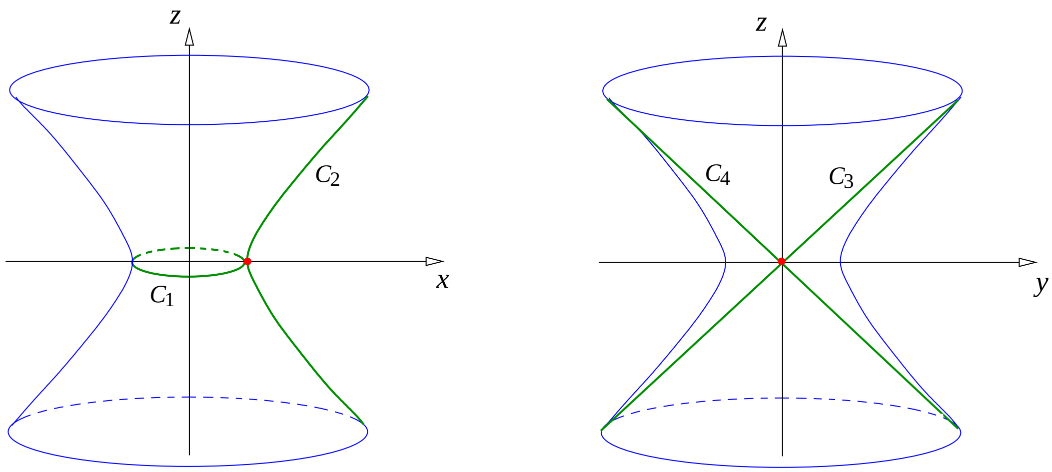
1. The hyperpoloid is generated by rotating the hyperpola $x^2 - z^2 = 1$ in the $x$-$z$ plane about the $z$-axis. We know that any normal section on the hyperbolid is a geodesic. Thus, the circle $C_1$: $x^2 + y^2 = 1$, $z = 0$ is a geodesic, as shown in part (a) of the figure. Half of the hyperbola, $C_2$: $x = \sqrt{z^2+1}$, i.e., the profile curve of the hyperboloid, is another geodesic.




(a) (b)

We also know that straight lines on a surface are geodesics. There are two such lines $C_3$ and $C_4$ through $(1,0,0)$, as shown in part (b) of the figure, with parametrizations: $(1,0,0) + t(0,1,1)$ and $(1,0,0) + t(0,1,-1)$, respectively.

2. We first obtain the velocity of the curve as follows:

$$\begin{aligned}\dot{\boldsymbol{\gamma}} &= \boldsymbol{\sigma}_u \cdot \dot{u} + \boldsymbol{\sigma}_v \cdot \dot{v} \\ &= (\cos v, \sin v, 0)\dot{u} + (-u\sin v, u\cos v, 1)\dot{v} \\ &= (\dot{u}\cos v - u\dot{v}\sin v, \dot{u}\sin v + u\dot{v}\cos v, \dot{v}).\end{aligned}$$

Because the curve is unit-speed, we have

$$\begin{aligned}1 &= \dot{\boldsymbol{\gamma}} \cdot \dot{\boldsymbol{\gamma}} \\ &= (\dot{u}\cos v - u\dot{v}\sin v)^2 + (\dot{u}\sin v + u\dot{v}\cos v)^2 + \dot{v}^2 \\ &= \dot{u}^2 + (1+u^2)\dot{v}^2. \qquad (1)\end{aligned}$$

Suppose $\boldsymbol{\gamma}$ is also a geodesic. Then the following geodesic equation must hold:

$$\frac{d}{dt}(F\dot{u} + G\dot{v}) = \frac{1}{2}(E_v\dot{u}^2 + 2F_v\dot{u}\dot{v} + G_v\dot{v}^2),$$

where $E, F, G$ are the coefficients of the first fundamental form of the surface. They are given as

$$\begin{aligned} E &= \boldsymbol{\sigma}_u \cdot \boldsymbol{\sigma}_u = 1, \\ F &= \boldsymbol{\sigma}_u \cdot \boldsymbol{\sigma}_v = 0, \\ G &= \boldsymbol{\sigma}_v \cdot \boldsymbol{\sigma}_v = 1 + u^2. \end{aligned}$$

Next, we obtain the partial derivatives:

$$E_v = F_v = G_v = 0.$$

Substitute the coefficients and their partial derivatives into the geodesic equation:

$$\frac{d}{dt}((1+u^2)\dot{v}) = 0.$$

Hence we have shown that

$$\dot{v} = \frac{a}{1+u^2}, \qquad \text{for constant } a.$$

(a) When $a = 0$, we have $\dot{v} = 0$. Hence $v = b$, for some constant $b$. Condition (1) reduces to $\dot{u}^2 = 1$. So we can use $u$ as the parameter. The geodesic is $\boldsymbol{\alpha}(u) = (u \cos b, u \sin b, b)$.

(b) When $a = 1$. Then

$$\dot{v} = \frac{1}{(1+u^2)}.$$

Plugging it into condition (1), we obtain that

$$\dot{u}^2 = \frac{u^2}{1+u^2} \qquad \dot{u} = \pm\frac{u}{1+u^2}.$$

We need only be concerned with the '+' sign which can be realized by choosing a direction of the unit-speed curve parametrization. Combining the two derivatives with respect to $t$, we obtain

$$\frac{dv}{du} = \frac{1}{u\sqrt{1+u^2}},$$

and

$$v = \int \frac{1}{u\sqrt{1+u^2}}\, du.$$

The curve is now parameterized in $u$ as

$$\boldsymbol{\beta}(u) = \left(u \cos\left(\int \frac{1}{u\sqrt{1+u^2}}\, du\right), u \sin\left(\int \frac{1}{u\sqrt{1+u^2}}\, du\right), \int \frac{1}{u\sqrt{1+u^2}}\, du\right).$$

3. Please see the Assignments page for an example.

   The smallest root of $f(x)$ is 0.37763. Typically the Bisection method needs most iterations, and the Newton method needs the least. Since Newton and Secant methods do not guarantee to converge, bounding the number of iterations may become nececessary.

4. (a) (b) (c) Please see the Assignments page for examples.

   (d)

$$\begin{aligned} p(x) \cdot q(x) = {} & x^{15} - 70.1x^{14} - 167.6x^{13} + 11913.914x^{12} + 6948.3586x^{11} \\ & -517455.3264x^{10} - 30525.0918x^9 + 2758544.6236x^8 + 8363.8288x^7 - 3520085.4888x^6 \\ & -92284.7352x^5 + 190309.36x^4 - 50718.16x^3 - 282918.4x^2 + 552960x - 348160\ . \end{aligned}$$