Solution to Assignment 6

Com S 477/577

Oct 08, 2010

1. We obtain all first and second derivatives of the surface:

$$\begin{aligned}\boldsymbol{\sigma}_u &= (1,0,2u),\\ \boldsymbol{\sigma}_v &= (0,1,2v),\\ \boldsymbol{\sigma}_{uu} &= (0,0,2),\\ \boldsymbol{\sigma}_{uv} &= (0,0,0),\\ \boldsymbol{\sigma}_{vv} &= (0,0,2),\end{aligned}$$

as well as the cross product

$$\boldsymbol{\sigma}_u \times \boldsymbol{\sigma}_v = (-2u,-2v,1),$$

which yields the surface normal

$$\boldsymbol{n} = \frac{(-2u,-2v,1)}{\sqrt{4u^2+4v^2+1}}.$$

The coefficients of the second fundamental form thus are

$$\begin{aligned}L &= \boldsymbol{\sigma}_{uu}\cdot\boldsymbol{n} = \frac{2}{\sqrt{4u^2+4v^2+1}},\\ M &= \boldsymbol{\sigma}_{uv}\cdot\boldsymbol{n} = 0,\\ N &= \boldsymbol{\sigma}_{vv}\cdot\boldsymbol{n} = \frac{2}{\sqrt{4u^2+4v^2+1}}.\end{aligned}$$

The second fundamental form is

$$\frac{2}{\sqrt{4u^2+4v^2+1}}\,(du^2+dv^2).$$

2. Using the dot '·' to denote differentiation with respect to u, we first obtain the two partial derivatives

$$\boldsymbol{\sigma}_u = (\dot{f}\cos v, \dot{f}\sin v, \dot{g}), \qquad \boldsymbol{\sigma}_v = (-f\sin v, f\cos v, 0),$$

and from them, the surface normal

$$\begin{aligned}N &= \frac{\boldsymbol{\sigma}_u\times\boldsymbol{\sigma}_v}{\|\boldsymbol{\sigma}_u\times\boldsymbol{\sigma}_v\|}\\ &= \frac{(-f\dot{g}\cos v, -f\dot{g}\sin v, \dot{f}f)}{f}, \qquad \text{since } \dot{f}^2+\dot{g}^2=1 \text{ due to the unit speed,}\\ &= (-\dot{g}\cos v, -\dot{g}\sin v, \dot{f}), \qquad \text{assuming } |f|>0.\end{aligned}$$

Along a meridian, $v = v_0$ for some constant v_0. Denote the meridian as $\boldsymbol{\alpha}(u) = \boldsymbol{\sigma}(u, v_0)$. Note that this is a unit-speed curve with $\dot{\boldsymbol{\alpha}} = \boldsymbol{\sigma}_u$, and

$$\ddot{\boldsymbol{\alpha}} = \boldsymbol{\sigma}_{uu} = (\ddot{f}\cos v, \ddot{f}\sin v, \ddot{g}).$$

Meanwhile,

$$\begin{aligned} N \times \dot{\boldsymbol{\alpha}} &= N \times \boldsymbol{\sigma}_u = \Big(-(\dot{f}^2 + \dot{g}^2)\sin v, (\dot{f}^2 + \dot{g}^2)\cos v, 0\Big) \\ &= (-\sin v, \cos v, 0), \end{aligned}$$

which gives the geodesic curvature

$$\ddot{\boldsymbol{\alpha}} \cdot (N \times \dot{\boldsymbol{\alpha}}) = 0.$$

Similarly, a parallel is denoted by $\boldsymbol{\beta}(v) = \boldsymbol{\sigma}(u_0, v)$ for some u_0. Since $\|\beta'(v)\| = \|\boldsymbol{\sigma}_v\| = f(u_0)$, the curve $\boldsymbol{\beta}$ is not unit-speed in v. Let $\boldsymbol{\gamma}(s)$ be the unit-speed reparametrization of β with $s = vf(u_0)$. We have

$$\boldsymbol{\gamma}'(s) = \frac{\boldsymbol{\sigma}_v}{\|\boldsymbol{\sigma}_v\|} = (-\sin v, \cos v, 0),$$

and

$$\boldsymbol{\gamma}''(s) = \frac{d}{dv}(\boldsymbol{\gamma}'(s)) \cdot \frac{dv}{ds} = \frac{1}{f}(-\cos v, -\sin v, 0).$$

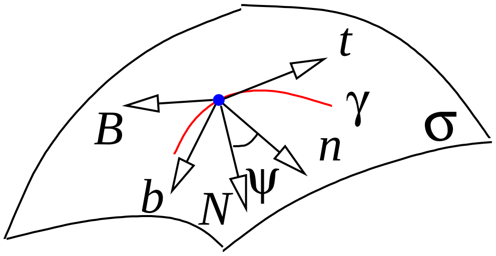
We also have

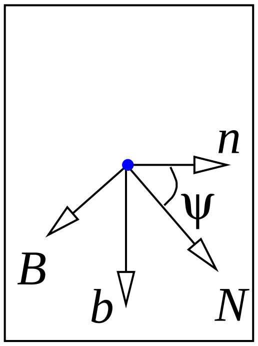
$$N \times \boldsymbol{\gamma}'(s) = (-\dot{f}\cos v, -\dot{f}\sin v, -\dot{g}).$$

Thus the geodesic curvature of the parallel $\boldsymbol{\gamma}(s)$, i.e., $\boldsymbol{\beta}(v)$ is

$$\boldsymbol{\gamma}''(s) \cdot \Big(N \times \boldsymbol{\gamma}'(s)\Big) = \frac{1}{f}(-\dot{f}\cos^2 v - \dot{f}\sin^2 v) = -\frac{\dot{f}}{f}.$$

3. Since the patch has normal curvature $\kappa > 0$, the angle ψ between $\ddot{\boldsymbol{\gamma}}$ and N must be in $[0, \frac{\pi}{2})$. The figure below illustrates the surface patch $\boldsymbol{\sigma}$ and the plane perpendicular to the curve tangent at a surface point on $\boldsymbol{\gamma}$.





The vectors $N, B, \boldsymbol{n}, \boldsymbol{b}$ are all orthogonal to $\boldsymbol{t}$ and thus must be coplanar. From the figures we have that

$$\begin{aligned} N &= (N \cdot \boldsymbol{n})\boldsymbol{n} + (N \cdot \boldsymbol{b})\boldsymbol{b} \\ &= \boldsymbol{n}\cos\psi + \boldsymbol{b}\sin\left(\frac{\pi}{2} - \psi\right) \\ &= \boldsymbol{n}\cos\psi + \boldsymbol{b}\sin\psi, \\ B &= (B \cdot \boldsymbol{n})\boldsymbol{n} + (B \cdot \boldsymbol{b})\boldsymbol{b} \\ &= \boldsymbol{n}\cos\left(\psi + \frac{\pi}{2}\right) + \boldsymbol{b}\cos\psi \\ &= \boldsymbol{b}\cos\psi - \boldsymbol{n}\sin\psi. \end{aligned}$$

Since $\boldsymbol{t} = \dot{\boldsymbol{\gamma}}$, $\dot{\boldsymbol{t}} = \ddot{\boldsymbol{\gamma}}$. Also, since $\boldsymbol{t}$ is a unit vector, $\boldsymbol{t} \cdot \dot{\boldsymbol{t}} = \frac{1}{2} d(\boldsymbol{t} \cdot \boldsymbol{t})/dt = 0$. Therefore, $\dot{\boldsymbol{t}}$ must be a linear combination of N and B. Project this derivative onto N and B:

$$\begin{aligned} \dot{t} \cdot N &= \ddot{\boldsymbol{\gamma}} \cdot N \\ &= \kappa_n, \\ \dot{t} \cdot B &= \ddot{\boldsymbol{\gamma}} \cdot (\boldsymbol{t} \times N) \\ &= -\kappa_g. \end{aligned}$$

Similarly, we project the derivative of N, orthogonal to N, onto $\boldsymbol{t}$ and B as follows:

$$\begin{aligned} \dot{N} \cdot \boldsymbol{t} &= -\boldsymbol{t} \cdot N, \qquad \text{since } d(\boldsymbol{t} \cdot N)/dt = 0, \\ &= -\kappa_n, \\ \dot{N} \cdot B &= \frac{d}{dt}(\boldsymbol{n} \cos\psi + \boldsymbol{b} \sin\psi) \cdot (\boldsymbol{b} \cos\psi - \boldsymbol{n} \sin\psi) \\ &= \Big(\dot{\boldsymbol{n}} \cos\psi - \boldsymbol{n}\dot{\psi} \sin\psi + \dot{\boldsymbol{b}} \sin\psi + \boldsymbol{b}\dot{\psi} \cos\psi\Big) \cdot (\boldsymbol{b} \cos\psi - \boldsymbol{n} \sin\psi) \\ &= \Big((-\kappa\boldsymbol{t} + \tau\boldsymbol{b}) \cos\psi + (-\tau\boldsymbol{n}) \sin\psi - \boldsymbol{n}\dot{\psi} \sin\psi + \boldsymbol{b}\dot{\psi} \cos\psi\Big) \cdot (\boldsymbol{b} \cos\psi - \boldsymbol{n} \sin\psi) \\ &= \tau(\cos^2\psi + \sin^2\psi) + \dot{\psi}(\cos^2\psi + \sin^2\psi) \\ &= \tau + \dot{\psi}. \end{aligned}$$

Note we made use of the Frenet formulae while differentiating $\boldsymbol{n}$ and $\boldsymbol{b}$ above. Letting $\tau_g = \tau + \dot{\psi}$, we have proved the expression for $\dot{N}$.

The derivative $\dot{B}$ is spanned by $\boldsymbol{t}$ and N. The expression for $\dot{B}$ follows from those for $\dot{\boldsymbol{t}}$ and $\dot{N}$ and the following identities:

$$\dot{B} \cdot N = -\dot{N} \cdot B \qquad \text{and} \qquad \dot{B} \cdot \boldsymbol{t} = -\dot{\boldsymbol{t}} \cdot B.$$

4. Reparametrize $\boldsymbol{\gamma}(t)$ as $\tilde{\boldsymbol{\gamma}}(s)$ using its arc length s. So we have

$$\dot{\boldsymbol{\gamma}} = \dot{\tilde{\boldsymbol{\gamma}}} \frac{ds}{dt}. \tag{1}$$

In the above, '$\cdot$' denotes differentiation with respect to the parameter, i.e., t of $\boldsymbol{\gamma}$ and s of $\tilde{\boldsymbol{\gamma}}$. Differentiate (1) once more:

$$\ddot{\boldsymbol{\gamma}} = \ddot{\tilde{\boldsymbol{\gamma}}} \left(\frac{ds}{dt}\right)^2 + \dot{\tilde{\boldsymbol{\gamma}}} \cdot \frac{d^2 s}{dt^2}.$$

Take the dot products of $\boldsymbol{n}$ with both sides of the above equation:

$$\begin{aligned} \ddot{\boldsymbol{\gamma}} \cdot \boldsymbol{n} &= (\ddot{\tilde{\boldsymbol{\gamma}}} \cdot \boldsymbol{n}) \left(\frac{ds}{dt}\right)^2 + (\dot{\tilde{\boldsymbol{\gamma}}} \cdot \boldsymbol{n}) \frac{d^2 s}{dt^2} \\ &= (\ddot{\tilde{\boldsymbol{\gamma}}} \cdot \boldsymbol{n}) \left(\frac{ds}{dt}\right)^2, \end{aligned}$$

where the last step follows from that the tangent $\dot{\tilde{\boldsymbol{\gamma}}}$ is orthogonal to the normal $\boldsymbol{n}$. Hence, the normal curvature is

$$\kappa_n = \ddot{\tilde{\boldsymbol{\gamma}}} \cdot \boldsymbol{n} = \frac{\ddot{\boldsymbol{\gamma}} \cdot \boldsymbol{n}}{\left(\frac{ds}{dt}\right)^2}. \tag{2}$$

It follows that

$$\begin{aligned} \left(\frac{ds}{dt}\right)^2 &= \dot{\boldsymbol{\gamma}} \cdot \dot{\boldsymbol{\gamma}} \\ &= (\boldsymbol{\sigma}_u \dot{u} + \boldsymbol{\sigma}_v \dot{v}) \cdot (\boldsymbol{\sigma}_u \dot{u} + \boldsymbol{\sigma}_v \dot{v}) \\ &= E\dot{u}^2 + 2F\dot{u}\dot{v} + G\dot{v}^2, \end{aligned} \tag{3}$$

where E, F, G are the coefficients of the first fundamental. Meanwhile, since

$$\ddot{\gamma} = \boldsymbol{\sigma}_{uu}\dot{u}^2 + 2\boldsymbol{\sigma}_{uv}\dot{u}\dot{v} + \boldsymbol{\sigma}_{vv}\dot{v}^2,$$

we obtain

$$\ddot{\gamma} \cdot \boldsymbol{n} = L\dot{u}^2 + 2M\dot{u}\dot{v} + N\dot{v}^2, \tag{4}$$

where L, M, N are the coefficients of the second fundamental form. Subtitution of (3) and (4) into (2) yields

$$\kappa_n = \frac{L\dot{u}^2 + 2M\dot{u}\dot{v} + N\dot{v}^2}{E\dot{u}^2 + 2F\dot{u}\dot{v} + G\dot{v}^2}.$$

5. We start with evaluating the partial derivatives of $\boldsymbol{\sigma}$ up to the second order

$$\begin{aligned} \boldsymbol{\sigma}_u &= (\cos v, \sin v, 0), \\ \boldsymbol{\sigma}_v &= (-u\sin v, u\cos v, b), \\ \boldsymbol{\sigma}_{uu} &= (0, 0, 0), \\ \boldsymbol{\sigma}_{uv} &= (-\sin v, \cos v, 0), \\ \boldsymbol{\sigma}_{vv} &= (-u\cos v, -u\sin v, 0). \end{aligned}$$

Next, we obtain the unit normal:

$$\begin{aligned} \boldsymbol{\sigma}_u \times \boldsymbol{\sigma}_v &= (b\sin v, -b\cos v, u), \\ \boldsymbol{n} &= \frac{(b\sin v, -b\cos v, u)}{\sqrt{u^2 + b^2}}. \end{aligned}$$

The first derivatives yield the coefficients of the first fundamental form:

$$E = 1, \qquad F = 0, \qquad G = u^2 + b^2,$$

and the second derivatives and $\boldsymbol{n}$ yield the coefficients of the second fundamental form:

$$L = 0, \qquad M = -\frac{b}{\sqrt{u^2 + b^2}}, \qquad N = 0.$$

Thus the Gaussian and mean curvatures:

$$\begin{aligned} K &= \frac{LN - M^2}{EG - F^2} &= -\frac{b^2}{(u^2 + b^2)^2}, \\ H &= \frac{LG - 2MF + NE}{2(EG - F^2)} &= 0. \end{aligned}$$

The principal curvatures are the roots of $\kappa^2 - 2H\kappa + K = 0$, i.e., of

$$\kappa^2 - \frac{b^2}{(u^2 + b^2)^2} = 0.$$

Their values are

$$\pm\frac{b}{u^2 + b^2}.$$

6. (**a**) The Gauss map is the equator of the unit sphere.

(**b**) Assuming that the standard unit normal point outward, the Gauss map is the circle in the southern hemisphere with lattitude $-\frac{\pi}{4}$.

(**c**) The point $(\frac{\sqrt{3}}{3}, \frac{\sqrt{3}}{3}, \frac{\sqrt{3}}{3})$.

(**d**) The entire unit sphere.