Solution to Assignment 4

Com S 477/577

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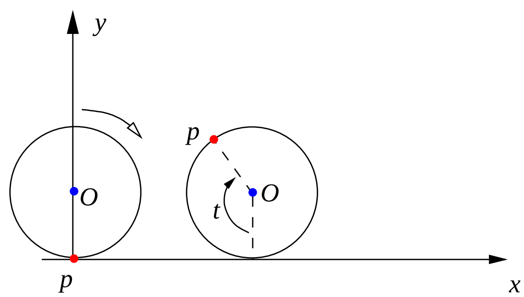
1. As shown in the figure below, let p be the initial contact point on the circle which has center O. We place the origin at the position of p before the rolling starts and let the straight line be the x-axis pointing in the moving direction of O.



As the circle rolls, the trajectory of p is a cycloid. Let t be the angle of rotation of the radius $\overline{Op}$ from its initial vertical position. Due to the rolling contact, rt is the distance traveled by the circle on the x-axis. Hence the cycloid has the parametric form

$$\begin{aligned}\alpha(t) &= (rt, r) + (r\cos(\frac{3\pi}{2} - t), r\sin(\frac{3\pi}{2} - t)) \\ &= (rt, r) - r(\sin t, \cos t) \\ &= r(t - \sin t, 1 - \cos t)\ .\end{aligned}$$

Now, we calculate the arc length along the cycloid after one complete revolution of the circle.

$$\begin{aligned}\alpha'(t) &= r(1 - \cos t, \sin t)\ , \\ \|\alpha'(t)\| &= r\sqrt{1 - 2\cos t + \cos^2 t + \sin^2 t} \\ &= r\sqrt{2(1 - \cos t)} \\ &= r\sqrt{2\left(1 - \left(1 - 2\sin^2 \frac{t}{2}\right)\right)} \\ &= r\sqrt{4\sin^2 \frac{t}{2}}, \\ &= 2r\left|\sin \frac{t}{2}\right|.\end{aligned}$$

Arc length $s(t)$ will be as follows:

$$\begin{aligned} s(t) &= \int_0^{2\pi} \|\alpha'(t)\|\, dt \\ &= \int_0^{2\pi} 2r \cdot \sin\frac{t}{2}\, dt \\ &= \left[-4r \cdot \cos\frac{t}{2}\right]_0^{2\pi} \\ &= 8r. \end{aligned}$$

2.

$$\begin{aligned} \widetilde{\gamma}(\widetilde{t}) &= \gamma(\phi(\widetilde{t})), \\ \widetilde{s} &= \int_{\widetilde{t}_0}^{\widetilde{t}} \|\widetilde{\gamma}'(\widetilde{t})\|\, d\widetilde{t}, \\ \widetilde{\gamma}'(\widetilde{t}) &= \gamma'(\phi(\widetilde{t}))\frac{d\phi}{d\widetilde{t}}, \\ \|\widetilde{\gamma}'(\widetilde{t})\| &= \|\gamma'(\phi(\widetilde{t}))\| \left|\frac{d\phi}{d\widetilde{t}}\right|, \end{aligned}$$

$$\begin{aligned} \widetilde{s} &= \int_{\widetilde{t}_0}^{\widetilde{t}} \|\widetilde{\gamma}'(\widetilde{t})\|\, d\widetilde{t} \\ &= \int_{\widetilde{t}_0}^{\widetilde{t}} \|\gamma'(\phi(\widetilde{t}))\frac{d\phi}{d\widetilde{t}}\|\, d\widetilde{t}, \\ &= \int_{\widetilde{t}_0}^{\widetilde{t}} \|\gamma'(\phi(\widetilde{t}))\| \left|\frac{d\phi}{d\widetilde{t}}\right|\, d\widetilde{t}. \end{aligned}$$

If $\frac{d\phi}{d\widetilde{t}} > 0$,

$$\begin{aligned} \widetilde{s} &= \int_{\widetilde{t}_0}^{\widetilde{t}} \|\gamma'(\phi(\widetilde{t}))\| \cdot \frac{d\phi}{d\widetilde{t}}\, d\widetilde{t} \\ &= \int_{t_0}^{t} \|\gamma'(\phi(\widetilde{t}))\|\, dt \\ &= s. \end{aligned}$$

If $\frac{d\phi}{d\widetilde{t}} < 0$,

$$\begin{aligned} \widetilde{s} &= \int_{\widetilde{t}_0}^{\widetilde{t}} \|\gamma'(\phi(\widetilde{t}))\| \cdot \left(-\frac{d\phi}{d\widetilde{t}}\right)\, d\widetilde{t} \\ &= \int_{t_0}^{t} -\|\gamma'(\phi(\widetilde{t}))\|\, dt \\ &= -s. \end{aligned}$$

3. (a) From $\alpha(\phi) = (a\cos\phi, b\sin\phi)$, let $x = a\cos\phi$ and $y = b\sin\phi$. We have

$$\begin{aligned}
T &= \frac{(-a\sin\phi, b\cos\phi)}{\sqrt{a^2\sin^2\phi + b^2\cos^2\phi}}, \\
N &= \frac{(-b\cos\phi, -a\sin\phi)}{\sqrt{a^2\sin^2\phi + b^2\cos^2\phi}}, \\
\kappa(\phi) &= \frac{x'y'' - x''y'}{(x'^2 + y'^2)^{\frac{3}{2}}} \\
&= \frac{(-a\sin\phi)(-b\sin\phi) - (-a\cos\phi)(b\cos\phi)}{((-a\sin\phi)^2 + (b\cos\phi)^2)^{\frac{3}{2}}} \\
&= \frac{ab\sin^2\phi + ab\cos^2\phi}{\left(a^2\sin^2\phi + b^2\cos^2\phi\right)^{\frac{3}{2}}} \\
&= \frac{ab}{\left(a^2\sin^2\phi + b^2\cos^2\phi\right)^{\frac{3}{2}}}.
\end{aligned}$$

The radius of curvature is $(a^2\sin^2\phi + b^2\cos^2\phi)^{3/2}/|ab|$, and the center of curvature is

$$\begin{aligned}
\alpha(\phi) + N/\kappa &= (a\cos\phi, b\sin\phi) + \frac{(a^2\sin^2\phi + b^2\cos^2\phi)^{\frac{3}{2}}}{ab} \cdot \frac{(-b\cos\phi, -a\sin\phi)}{(a^2\sin^2\phi + b^2\cos^2\phi)^{\frac{1}{2}}} \\
&= (a\cos\phi, b\sin\phi) + (a^2\sin^2\phi + b^2\cos^2\phi)\left(-\frac{\cos\phi}{a}, -\frac{\sin\phi}{b}\right) \\
&= \left(\frac{a^2\cos\phi - a^2\sin^2\phi\cos\phi - b^2\cos^3\phi}{a}, \frac{b^2\sin\phi - a^2\sin^3\phi - b^2\sin\phi\cos^2\phi}{b}\right) \\
&= \left(\frac{a^2 - b^2}{a}\cos^3\phi, \frac{b^2 - a^2}{b}\sin^3\phi\right).
\end{aligned}$$

(b) Solution 1:

From $\beta(t) = \left(\frac{t^2}{1+t^2}, \frac{t^3}{1+t^2}\right)$, let x, y, x', y', x'' and y'' as follows:

$$\begin{aligned}
x &= \frac{t^2}{1+t^2}, & y &= \frac{t^3}{1+t^2}; \\
x' &= \frac{2t}{(1+t^2)^2}, & y' &= \frac{t^2(3+t^2)}{(1+t^2)^2}; \\
x'' &= \frac{2(1-3t^2)}{(1+t^2)^3}, & y'' &= \frac{2t(3-t^2)}{(1+t^2)^3}.
\end{aligned}$$

Center of the curvature is calculated based on the following formula:

$$\beta(t) + \frac{1}{\kappa(t)} N(t), \text{ where}$$

$$\begin{aligned}
x'^2 + y'^2 &= \frac{t^2(t^2+4)}{(t^2+1)^2}, \\
N(t) &= \frac{(-y', x')}{\sqrt{x'^2 + y'^2}} \\
&= \left(-\frac{t^2(t^2+3)}{(t^2+1)\sqrt{t^4+4t^2}}, \frac{2t}{(t^2+1)\sqrt{t^4+4t^2}} \right), \\
\kappa(t) &= \frac{x'y'' - x''y'}{(x'^2+y'^2)^{\frac{3}{2}}} \\
&= \frac{-4t^2(t^2-3) + 2t^2(t^2+3)(3t^2-1)}{(t^2+1)^2(t^4+4t^2)^{\frac{3}{2}}} \\
&= \frac{6t^2}{(t^4+4t^2)^{\frac{3}{2}}} \\
&= \frac{6}{|t|(t^2+4)},
\end{aligned}$$

$$\begin{aligned}
\beta(t) + \frac{1}{\kappa(t)} N(t) &= \left(\frac{t^2}{1+t^2}, \frac{t^3}{1+t^2} \right) \\
&\quad + \frac{(t^4+4t^2)^{\frac{3}{2}}}{6t^2} \cdot \left(-\frac{t^2(t^2+3)}{(t^2+1)\sqrt{t^4+4t^2}}, \frac{2t}{(t^2+1)\sqrt{t^4+4t^2}} \right) \\
&= \left(-\frac{t^2(t^2+6)}{6}, \frac{4t}{3} \right).
\end{aligned}$$

Solution 2 (courtesy Samajit Das). Letting $t = \tan\theta$, the curve is reparametrized as

$$\alpha(\theta) = \beta(t) = (\sin^2\theta, \tan\theta \sin^2\theta).$$

The curvature, radius of curvature, and center of curvature are invariant to parametrization. We compute them in terms of θ, and substitute t back in the end. Less effort is involved in the calculation. :-)

4. We first differentiate the curve:

$$\beta'(s) = \left(\frac{1}{2}\sqrt{1+s}, -\frac{1}{2}\sqrt{1-s}, \frac{1}{\sqrt{2}} \right),$$

and obtain the speed:

$$\|\beta'(s)\| = \sqrt{\frac{1}{4}(1+s) + \frac{1}{4}(1-s) + \frac{1}{2}} = 1.$$

Thus the curve is unit speed and $T = \beta'(s)$. We obtain the second derivative of β, the curvature, and the principal normal:

$$\begin{aligned}
\beta''(s) &= \left(\frac{1}{4\sqrt{1+s}}, \frac{1}{4\sqrt{1-s}}, 0\right), \\
\kappa(s) &= \|\beta''\| \\
&= \frac{1}{2\sqrt{2(1-s^2)}}, \\
N &= \frac{\beta''(s)}{\kappa(s)} \\
&= \frac{\sqrt{2}}{2}(\sqrt{1-s}, \sqrt{1+s}, 0).
\end{aligned}$$

Next, the binormal is

$$\begin{aligned}
B &= T \times N \\
&= \beta'(s) \times N \\
&= \left(\frac{1}{2}\sqrt{1+s}, -\frac{1}{2}\sqrt{1-s}, \frac{1}{\sqrt{2}}\right) \times \frac{\sqrt{2}}{2}(\sqrt{1-s}, \sqrt{1+s}, 0) \\
&= \frac{1}{2}(-\sqrt{1+s}, \sqrt{1-s}, \sqrt{2}).
\end{aligned}$$

Since β is unit speed, $B' = -\kappa N$. We have

$$\begin{aligned}
B' &= \frac{1}{2}\left(-\frac{1}{2\sqrt{1+s}}, -\frac{1}{2\sqrt{1-s}}, 0\right), \\
&= \frac{1}{4}\left(-\frac{1}{\sqrt{1+s}}, -\frac{1}{\sqrt{1-s}}, 0\right) \\
&= -\frac{1}{2\sqrt{2(1-s^2)}}N.
\end{aligned}$$

Comparing the two forms of B' above, we have

$$\tau(s) = \frac{1}{2\sqrt{2(1-s^2)}}.$$

5. Since the curve lies on the sphere, we immediately have

$$(\alpha - \boldsymbol{c}) \cdot (\alpha - \boldsymbol{c}) = r^2.$$

Differentiating the above equation yields

$$T \cdot (\alpha - \boldsymbol{c}) = 0, \tag{1}$$

where $T = \alpha'$ is the unit tangent of the curve. The vector $\alpha - \boldsymbol{c}$ is represented in the frame defined by T, N, B as

$$\alpha - \boldsymbol{c} = ((\alpha - \boldsymbol{c}) \cdot N)N + ((\alpha - \boldsymbol{c}) \cdot B)B. \tag{2}$$

Thus we need to determine its dot products with N and B.

Differentiate equation (1):

$$\kappa N \cdot (\alpha - \boldsymbol{c}) + T \cdot T = 0,$$

from which we obtain the dot product with N:

$$N \cdot (\alpha - \boldsymbol{c}) = -\frac{1}{\kappa} = -\rho.$$

One more round of differentiation yields

$$\tau B \cdot (\alpha - \boldsymbol{c}) + N \cdot T = -\rho',$$

hence

$$(\alpha - \boldsymbol{c}) \cdot B = -\frac{\rho'}{\tau} = -\rho'\sigma.$$

Substitution of the two dot products involving $\alpha - \boldsymbol{c}$ into (2) proves that

$$\alpha - \boldsymbol{c} = -\rho N - \rho'\sigma B.$$

Finally, we take dot products of both sides of the above equation with themselves:

$$r^2 = (\alpha - \boldsymbol{c}) \cdot (\alpha - \boldsymbol{c}) = \rho^2 + (\rho'\sigma)^2.$$