Solution to Assignment 3

Com S 477/577

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1. According to the lecture notes, the orthographic projection matrix is

$$M = \boldsymbol{v}\boldsymbol{n}^T - (\boldsymbol{v}\cdot\boldsymbol{n})I_4 = \begin{pmatrix} n_2^2+n_3^2 & -n_1n_2 & -n_1n_3 & -n_1n_4 \\ -n_1n_2 & n_1^2+n_3^2 & -n_2n_3 & -n_2n_4 \\ -n_1n_3 & -n_2n_3 & n_1^2+n_2^2 & -n_3n_4 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

To get foreshortening ratios in the x-, y-, and z-directions, we multiply the transformation to a unit vector in each direction to get the projected length.

In the x-direction, we get the foreshortening ratio

$$\left\| M \circ \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \right\| = \left\| \begin{pmatrix} n_2^3+n_3^2 \\ -n_1n_2 \\ -n_1n_3 \\ 0 \end{pmatrix} \right\| = \sqrt{(n_2^3+n_3^2)^2+(n_1n_2)^2+(n_1n_3)^2} = \sqrt{(n_1^2+n_2^2+n_3^2)(n_2^2+n_3^2)}$$

$$= \sqrt{(n_2^2+n_3^2)}. \qquad (\text{ since } n_1^2+n_2^2+n_3^2=1 \text{ })$$

Similarly, in the y- and z- directions, we multiply the projection matrix to $(0,1,0,0)$ and $(0,0,1,0)$, obtaining the foreshortening ratios $(n_1^2+n_3^2)^{1/2}$ and $(n_1^2+n_2^2)^{1/2}$ respectively.

2. We have the normal $\boldsymbol{n} = (n_1, n_2, n_3, n_4)$ of the projection and the viewpoint $\boldsymbol{v} = (v_1, v_2, v_3, v_4)$. The perspective projection matrix is

$$M = \boldsymbol{v}\boldsymbol{n}^T - (\boldsymbol{v}\cdot\boldsymbol{n})I_4$$

$$= \begin{pmatrix} -n_2v_2-n_3v_3-n_4v_4 & n_2v_1 & n_3v_1 & n_4v_1 \\ n_1v_2 & -n_1v_1-n_3v_3-n_4v_4 & n_3v_2 & n_4v_2 \\ n_1v_3 & n_2v_3 & -n_1v_1-n_2v_2-n_4v_4 & n_4v_3 \\ n_1v_4 & n_2v_4 & n_3v_4 & -n_1v_1-n_2v_2-n_3v_3 \end{pmatrix}.$$

The infinity point at direction (x, y, z) is projected to a vanishing point as follows:

$$M \circ \begin{pmatrix} x \\ y \\ z \\ 0 \end{pmatrix} = \begin{pmatrix} -x(n_2v_2+n_3v_3+n_4v_4)+yn_2v_1+zn_3v_1 \\ xn_1v_2-y(n_1v_1+n_3v_3+n_4v_4)+zn_3v_2 \\ xn_1v_3+yn_2v_3-z(n_1v_1+n_2v_2+n_4v_4) \\ (xn_1+yn_2+zn_3)v_4 \end{pmatrix},$$

which, in Cartesian coordinates, is (p_1, p_2, p_3), where

$$\begin{aligned}
p_1 &= \frac{-x(n_2v_2 + n_3v_3 + n_4v_4) + yn_2v_1 + zn_3v_1}{v_4(xn_1 + yn_2 + zn_3)},\\
p_2 &= \frac{xn_1v_2 - y(n_1v_1 + n_3v_3 + n_4v_4) + zn_3v_2}{v_4(xn_1 + yn_2 + zn_3)},\\
p_3 &= \frac{xn_1v_3 + yn_2v_3 - z(n_1v_1 + n_2v_2 + n_4v_4)}{v_4(xn_1 + yn_2 + zn_3)}.
\end{aligned}$$

3. (a) The projection matrix and the transformed vertices are

$$\begin{pmatrix} 5 & 6 & 4 & -8 \\ 1 & 4 & -2 & 4 \\ -1 & 3 & 9 & -4 \\ -1 & 3 & 2 & 3 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix} = \begin{pmatrix} -8 & -3 & -2 & 7 \\ 4 & 5 & 8 & 7 \\ -4 & -5 & -1 & 7 \\ 3 & 2 & 6 & 7 \end{pmatrix}.$$

(b)

$$\begin{pmatrix} -5 & 0 & -3 & 2 \\ 20 & -10 & -12 & 8 \\ -5 & 0 & -7 & -2 \\ 5 & 0 & -3 & -8 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix} = \begin{pmatrix} 2 & -3 & 2 & -6 \\ 8 & 28 & -2 & 6 \\ -2 & -7 & -2 & -14 \\ -8 & -3 & -8 & -6 \end{pmatrix}$$

(c) Note that it's a parallel projection:

$$\begin{pmatrix} -5 & 2 & 3 & 4 \\ 0 & -9 & -6 & -8 \\ 0 & 6 & 4 & 12 \\ 0 & 0 & 0 & -5 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix} = \begin{pmatrix} 4 & -1 & 6 & 4 \\ -8 & -8 & -17 & -23 \\ 12 & 12 & 18 & 22 \\ -5 & -5 & -5 & -5 \end{pmatrix}.$$

(d)

$$\begin{pmatrix} 32 & 0 & 0 & 0 \\ 28 & 0 & 0 & 20 \\ 63 & -72 & 32 & 45 \\ 0 & 0 & 0 & 32 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 32 & 0 & 32 \\ 20 & 48 & 20 & 48 \\ 45 & 108 & -27 & 68 \\ 32 & 32 & 32 & 32 \end{pmatrix}$$