

Solution to Assignment 2

Com S 477/577

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1. (a) The rotation matrix $\text{Rot}_{zyz}(\alpha, \beta, \gamma)$ is the concatenation of the three rotation matrix about axes $z_B, y_B,$ and $z_B,$ i.e., $\text{Rot}_z(\alpha) \circ \text{Rot}_y(\beta) \circ \text{Rot}_z(\gamma)$. Thus

$$\begin{aligned} \text{Rot}_{zyz}(\alpha, \beta, \gamma) &= \begin{pmatrix} \cos \alpha & -\sin \alpha & 0 \\ \sin \alpha & \cos \alpha & 0 \\ 0 & 0 & 1 \end{pmatrix} \circ \begin{pmatrix} \cos \beta & 0 & \sin \beta \\ 0 & 1 & 0 \\ -\sin \beta & 0 & \cos \beta \end{pmatrix} \circ \begin{pmatrix} \cos \gamma & -\sin \gamma & 0 \\ \sin \gamma & \cos \gamma & 0 \\ 0 & 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} \cos \alpha \cos \beta \cos \gamma - \sin \alpha \sin \gamma & -\cos \alpha \cos \beta \sin \gamma - \sin \alpha \cos \gamma & \cos \alpha \sin \beta \\ \sin \alpha \cos \beta \cos \gamma + \cos \alpha \sin \gamma & -\sin \alpha \cos \beta \sin \gamma + \cos \alpha \cos \gamma & \sin \alpha \sin \beta \\ -\sin \beta \cos \gamma & \sin \beta \sin \gamma & \cos \beta \end{pmatrix} \end{aligned}$$

Note that the three rotation angles are about the body frame instead of world frame. In space,

three rotations taken about fixed axes yield the same final orientation as the same three rotations taken in opposite order about the axes of the moving frame.

- (b) Given that

$$\text{Rot}_{zyz}(\alpha, \beta, \gamma) = \begin{pmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{pmatrix},$$

we consider the cases where $\sin \beta \neq 0$ and $\sin \beta = 0$ separately.

If $\sin \beta \neq 0$, we can assume that $\sin \beta > 0$. Denote by $\text{atan2}(c, s)$ the unique angle $\theta \in [0, \pi]$ such that $\cos \theta = c$ and $\sin \theta = s$. This function can easily be defined using \arctan . Then

$$\begin{aligned} \beta &= \text{atan2}(r_{33}, \sqrt{r_{31}^2 + r_{32}^2}), \\ \alpha &= \text{atan2}(r_{13}/\sin \beta, r_{23}/\sin \beta), \\ \gamma &= \text{atan2}(-r_{31}/\sin \beta, r_{32}/\sin \beta). \end{aligned}$$

There exists a second solution, but we only compute the single solution for which $0.0 \leq \beta \leq 180.0^\circ$ by using the positive square root in the formula for β .

If $\sin \beta = 0.0$, the above solution degenerates, and only the sum or the difference of α and γ may be computed.

If $\beta = 0.0^\circ$, then the rotation matrix becomes

$$\begin{pmatrix} \cos \alpha \cos \gamma - \sin \alpha \sin \gamma & -\cos \alpha \sin \gamma - \sin \alpha \cos \gamma & 0 \\ \sin \alpha \cos \gamma + \cos \alpha \sin \gamma & -\sin \alpha \sin \gamma + \cos \alpha \cos \gamma & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} \cos(\alpha + \gamma) & -\sin(\alpha + \gamma) & 0 \\ \sin(\alpha + \gamma) & \cos(\alpha + \gamma) & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

which gives

$$\alpha + \gamma = \text{atan2}(r_{11}, -r_{12}) .$$

If $\beta = 180.0^\circ$, then the rotation matrix becomes

$$\begin{pmatrix} -\cos \alpha \cos \gamma - \sin \alpha \sin \gamma & \cos \alpha \sin \gamma - \sin \alpha \cos \gamma & 0 \\ -\sin \alpha \cos \gamma + \cos \alpha \sin \gamma & \sin \alpha \sin \gamma + \cos \alpha \cos \gamma & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} -\cos(\alpha - \gamma) & -\sin(\alpha - \gamma) & 0 \\ -\sin(\alpha - \gamma) & \cos(\alpha - \gamma) & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

which gives

$$\alpha - \gamma = \text{atan2}(-r_{11}, -r_{12}) .$$

2. According to the lecture notes, the orthographic projection matrix is

$$M = \mathbf{v}\mathbf{n}^T - (\mathbf{v} \cdot \mathbf{n})I_4 = \begin{pmatrix} n_2^2 + n_3^2 & -n_1n_2 & -n_1n_3 & -n_1n_4 \\ -n_1n_2 & n_1^2 + n_3^2 & -n_2n_3 & -n_2n_4 \\ -n_1n_3 & -n_2n_3 & n_1^2 + n_2^2 & -n_3n_4 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

To get foreshortening ratios in the x -, y -, and z -directions, we multiply the transformation to a unit vector in each direction to get the projected length.

In x -direction, we get the foreshortening ratio

$$\begin{aligned} \left\| M \circ \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \right\| &= \left\| \begin{pmatrix} n_2^2 + n_3^2 \\ -n_1n_2 \\ -n_1n_3 \\ 0 \end{pmatrix} \right\| = \sqrt{(n_2^2 + n_3^2)^2 + (n_1n_2)^2 + (n_1n_3)^2} = \sqrt{(n_1^2 + n_2^2 + n_3^2)(n_2^2 + n_3^2)} \\ &= \sqrt{(n_2^2 + n_3^2)}. \quad (\text{since } n_1^2 + n_2^2 + n_3^2 = 1) \end{aligned}$$

Similarly, in y - and z - directions, we multiply the projection matrix to $(0, 1, 0, 0)$ and $(0, 0, 1, 0)$, and we get foreshortening ratios $(n_1^2 + n_3^2)^{1/2}$ and $(n_1^2 + n_2^2)^{1/2}$ respectively.

3. We have the normal $\mathbf{n} = (n_1, n_2, n_3, n_4)$ of the projection and the viewpoint $\mathbf{v} = (v_1, v_2, v_3, v_4)$. The perspective projection matrix is

$$M = \mathbf{v}\mathbf{n}^T - (\mathbf{v} \cdot \mathbf{n})I_4 = \begin{pmatrix} -n_2v_2 - n_3v_3 - n_4v_4 & n_2v_1 & n_3v_1 & n_4v_1 \\ n_1v_2 & -n_1v_1 - n_3v_3 - n_4v_4 & n_3v_2 & n_4v_2 \\ n_1v_3 & n_2v_3 & -n_1v_1 - n_2v_2 - n_4v_4 & n_4v_3 \\ n_1v_4 & n_2v_4 & n_3v_4 & -n_1v_1 - n_2v_2 - n_3v_3 \end{pmatrix}.$$

The vanishing point at direction (x, y, z) is then projected to

$$M \circ \begin{pmatrix} x \\ y \\ z \\ 0 \end{pmatrix} = \begin{pmatrix} -x(n_2v_2 + n_3v_3 + n_4v_4) + yn_2v_1 + zn_3v_1 \\ xn_1v_2 - y(n_1v_1 + n_3v_3 + n_4v_4) + zn_3v_2 \\ xn_1v_3 + yn_2v_3 - z(n_1v_1 + n_2v_2 + n_4v_4) \\ (xn_1 + yn_2 + zn_3)v_4 \end{pmatrix},$$

which is in homogeneous coordinates. Thus the point in Cartesian coordinates (p_1, p_2, p_3) is obtained by the corresponding homogeneous coordinates $(p_1, p_2, p_3, 1)$.

4. (a) The projection matrix and transformed vertices are

$$\begin{pmatrix} 5 & 6 & 4 & -8 \\ 1 & 4 & -2 & 4 \\ -1 & 3 & 9 & -4 \\ -1 & 3 & 2 & 3 \end{pmatrix} \begin{pmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} & \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} & \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix} & \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \end{pmatrix} = \begin{pmatrix} \begin{bmatrix} -8 \\ 4 \\ -4 \\ 3 \end{bmatrix} & \begin{bmatrix} -3 \\ 5 \\ -5 \\ 2 \end{bmatrix} & \begin{bmatrix} -2 \\ 8 \\ -1 \\ 6 \end{bmatrix} & \begin{bmatrix} 7 \\ 7 \\ 7 \\ 7 \end{bmatrix} \end{pmatrix}$$

(b)

$$\begin{pmatrix} -5 & 0 & -3 & 2 \\ 20 & -10 & -12 & 8 \\ -5 & 0 & -7 & -2 \\ 5 & 0 & -3 & -8 \end{pmatrix} \begin{pmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} & \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} & \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix} & \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \end{pmatrix} = \begin{pmatrix} \begin{bmatrix} 2 \\ 8 \\ -2 \\ -8 \end{bmatrix} & \begin{bmatrix} -3 \\ 28 \\ -7 \\ -3 \end{bmatrix} & \begin{bmatrix} 2 \\ -2 \\ -2 \\ -8 \end{bmatrix} & \begin{bmatrix} -6 \\ 6 \\ -14 \\ -6 \end{bmatrix} \end{pmatrix}$$

(c) Note that it's a parallel projection.

$$\begin{pmatrix} -5 & 2 & 3 & 4 \\ 0 & -9 & -6 & -8 \\ 0 & 6 & 4 & 12 \\ 0 & 0 & 0 & -5 \end{pmatrix} \begin{pmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} & \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} & \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix} & \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \end{pmatrix} = \begin{pmatrix} \begin{bmatrix} 4 \\ -8 \\ 12 \\ -5 \end{bmatrix} & \begin{bmatrix} -1 \\ -8 \\ 12 \\ -5 \end{bmatrix} & \begin{bmatrix} 6 \\ -17 \\ 18 \\ -5 \end{bmatrix} & \begin{bmatrix} 4 \\ -23 \\ 22 \\ -5 \end{bmatrix} \end{pmatrix}$$

(d)

$$\begin{pmatrix} 32 & 0 & 0 & 0 \\ 28 & 0 & 0 & 20 \\ 63 & -72 & 32 & 45 \\ 0 & 0 & 0 & 32 \end{pmatrix} \begin{pmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} & \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} & \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix} & \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \end{pmatrix} = \begin{pmatrix} \begin{bmatrix} 0 \\ 20 \\ 45 \\ 32 \end{bmatrix} & \begin{bmatrix} 32 \\ 48 \\ 108 \\ 32 \end{bmatrix} & \begin{bmatrix} 0 \\ 20 \\ -27 \\ 32 \end{bmatrix} & \begin{bmatrix} 32 \\ 48 \\ 68 \\ 32 \end{bmatrix} \end{pmatrix}$$