Solution to Assignment 10

Com S 477/577

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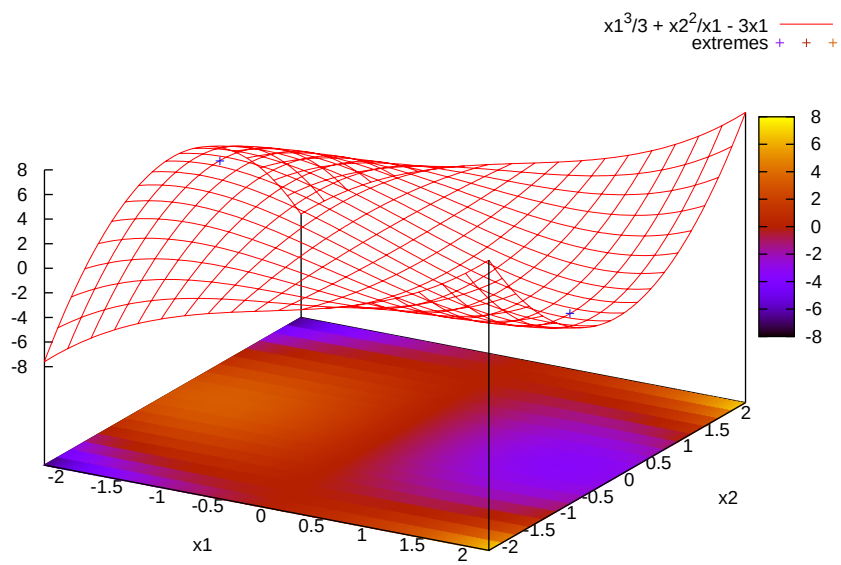
1. (courtesy Wen-Chieh Chang)

   Please see the Assignments page for an example.

   The function has a local minimum $-3.46410 = -2\sqrt{3}$ at $(\sqrt{3}, 0)$ and a local maximum $3.46410 = 2\sqrt{3}$ at $(-\sqrt{3}, 0)$.

   Note that the function $f(x_1, x_2)$ is a cubic polynomial on x_1 and x_2, hence the global maximum and global minimum are ∞ and $-\infty$. In other words, the start point of the search is important.



2. Apply the method of Lagrange multipliers. We find the Lagrangian and its gradient:

$$
\begin{aligned}
F(x_1, x_2, x_3, \lambda) &= f(x_1, x_2, x_3) + \lambda h(x_1, x_2, x_3) \\
&= (3x_1^2 + 4x_2^2 + 5x_3^2) + \lambda(x_1 + x_2 + x_3 - 10) \ , \\
\nabla F &= \begin{pmatrix} 6x_1 + \lambda \\ 8x_2 + \lambda \\ 10x_3 + \lambda \\ x_1 + x_2 + x_3 - 10 \end{pmatrix} = 0 \ .
\end{aligned}
$$

   Solving the above system yields the optimal solution $(x_1, x_2, x_3) = (\frac{200}{47}, \frac{150}{47}, \frac{120}{47})$ while $\lambda = -\frac{1200}{47}$.

3. An inscribed parallelepiped has its eight vertices on the ellipsoid. Let (x, y, z) denote one of these vertices, where x,y,$z > 0$. The volume of the parallelepiped is then $8xyz$. There fore we are to solve the following constrained optimization problem:

$$\begin{array}{ll} \text{maximize} & 8xyz \\ \text{subject to} & \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1. \end{array}$$

At a maximum point, the Lagrangian

$$F(x, y, z, \lambda) = 8xyz + \lambda \left(\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} - 1 \right)$$

must have vanishing partial derivatives with respect to x, y, z, and to the Lagrange multiplier λ (the first order necessary condition). This yields

$$\frac{\partial F}{\partial x} = 8yz + \lambda \frac{2x}{a^2} = 0; \tag{1}$$

$$\frac{\partial F}{\partial y} = 8xz + \lambda \frac{2y}{b^2} = 0; \tag{2}$$

$$\frac{\partial F}{\partial z} = 8xy + \lambda \frac{2z}{c^2} = 0. \tag{3}$$

Multiply (1) by x and (2) by y, and substract the second resulting equation from the first one. This yields

$$\lambda \left(\frac{2x^2}{a^2} - \frac{2y^2}{b^2} \right) = 0. \tag{4}$$

Similar steps of manipulation applied to the other two combinations of equations (1), (2), (3) yield

$$\lambda \left(\frac{2x^2}{a^2} - \frac{2y^2}{b^2} \right) = 0, \tag{5}$$

$$\lambda \left(\frac{2x^2}{a^2} - \frac{2y^2}{b^2} \right) = 0. \tag{6}$$

Equations (4)-(6) together imply that either $\lambda = 0$ or

$$\frac{x^2}{a^2} = \frac{y^2}{b^2} = \frac{z^2}{c^2}. \tag{7}$$

Obviously $\lambda \neq 0$ as otherwise at least two of x, y, z would have to be zero. Substituting equation (7) back into the equation of the ellipsoid, we obtain

$$x = \frac{a}{\sqrt{3}};$$

$$y = \frac{b}{\sqrt{3}};$$

$$z = \frac{c}{\sqrt{3}}.$$

Hence the maximum volume is $\frac{8abc}{3\sqrt{3}}$, which is achieved at points $(\pm\frac{a}{\sqrt{3}}, \pm\frac{b}{\sqrt{3}}, \pm\frac{c}{\sqrt{3}})$.