

Solution to Assignment 1

Com S 477/577

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1. The direction vector \mathbf{v} in the shear transformation is a unit vector of $(3, -4)$, hence $\mathbf{v} = (3/5, -4/5)$. According to the lecture notes, the shear transformation matrix with direction \mathbf{v} and factor 4 is

$$\begin{aligned}\text{Shear}(\mathbf{v}, 4) &= \begin{pmatrix} 1 - 4 \cdot \frac{3}{5} \cdot (-\frac{4}{5}) & 4 \cdot (\frac{3}{5})^2 \\ -4 \cdot (-\frac{4}{5})^2 & 1 + 4 \cdot \frac{3}{5} \cdot (-\frac{4}{5}) \end{pmatrix} \\ &= \begin{pmatrix} \frac{73}{25} & \frac{36}{25} \\ -\frac{64}{25} & -\frac{23}{25} \end{pmatrix} \\ &= \begin{pmatrix} 2.92 & 1.44 \\ -2.56 & -0.92 \end{pmatrix}.\end{aligned}$$

After applying the transformation to the vertices, the new coordinates are

$$\begin{pmatrix} 2.92 & 1.44 \\ -2.56 & -0.92 \end{pmatrix} \begin{pmatrix} 2 & 0 & -5 & 3 \\ 2 & 4 & 1 & -2 \end{pmatrix} = \begin{pmatrix} 8.72 & 5.76 & -13.16 & 5.88 \\ -6.96 & -3.68 & 11.88 & -5.84 \end{pmatrix}.$$

2. Let $ax + by + c = 0$ be a line l in the Cartesian plane and (u, v, w) a point on the line in homogeneous coordinates. Therefore $(\frac{u}{w}, \frac{v}{w})$ is the point on the line in the Cartesian plane, and for all homogeneous coordinates (u, v, w) on the line, we have

$$au + bv + cw = 0. \tag{1}$$

In the spherical model, a point with homogeneous coordinate (u, v, w) maps to the intersection of $t(u, v, w)$ and the unit sphere

$$u^2 + v^2 + w^2 = 1. \tag{2}$$

Thus a point on l must satisfy both (1) and (2). In a 3D space, since (1) represents a plane of normal (a, b, c) passing the origin $(0, 0, 0)$, which is the center of the unit sphere defined by (2), the intersection is the great circle described by the simultaneous equations with variables u, v , and w .

3. (a) The line ℓ passing through $(2, -4)$ and $(11, 13)$ satisfies equations

$$\begin{aligned}\ell \cdot (2, -4, 1) &= 0, \\ \ell \cdot (11, 13, 1) &= 0.\end{aligned}$$

Hence ℓ in the homogeneous coordinates can be specified by

$$\ell = (2, -4, 1) \times (11, 13, 1) = (-17, 9, 70).$$

In the Cartesian plane, ℓ has the equation $17x - 9y - 70 = 0$.

- (b) The intersection point \mathbf{p} of the lines $-2x + 3y - 8 = 0$ and $5x + 2y + 6 = 0$ satisfies

$$\begin{aligned}(-2, 3, -8) \cdot \mathbf{p} &= 0, \\ (5, 2, 6) \cdot \mathbf{p} &= 0.\end{aligned}$$

Hence in the homogeneous coordinates,

$$\mathbf{p} = (-2, 3, -8) \times (5, 2, 6) = (34, -28, -19),$$

and $\mathbf{p} = (-\frac{34}{19}, \frac{28}{19})$ in the Cartesian plane.

4. (a) By computing the determinant

$$\begin{vmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 & \mathbf{e}_4 \\ 2 & -1 & 1 & 0 \\ -4 & 3 & -2 & 5 \\ 1 & 1 & 0 & -6 \end{vmatrix} = 11\mathbf{e}_1 - 5\mathbf{e}_2 - 27\mathbf{e}_3 + \mathbf{e}_4,$$

we obtain the intersection point $(11, -5, -27)$ in \mathbb{R}^3 .

- (b) By computing the determinant

$$\begin{vmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 & \mathbf{e}_4 \\ 2 & -1 & 1 & 1 \\ -9 & 7 & 3 & 1 \\ 10 & 6 & -2 & 1 \end{vmatrix} = -38\mathbf{e}_1 - 17\mathbf{e}_2 - 141\mathbf{e}_3 + 200\mathbf{e}_4,$$

we obtain the plane $38x + 17y + 141z = 200$ containing the three points in \mathbb{R}^3 .

5. Following the lecture notes, the reflection matrix in the line $5x - 2y + 8 = 0$ is

$$\text{Ref}_{(5,-2,8)} = \begin{pmatrix} (-2)^2 - (5)^2 & -2(5)(-2) & -2(5)(8) \\ -2(5)(-2) & (5)^2 - (-2)^2 & -2(-2)(8) \\ 0 & 0 & (5)^2 + (-2)^2 \end{pmatrix} = \begin{pmatrix} -21 & 20 & -80 \\ 20 & 21 & 32 \\ 0 & 0 & 29 \end{pmatrix}.$$

The reflection of $(3, 2)$ in homogeneous coordinates is $\text{Ref}_{(5,-2,8)}(3, 2, 1)^T = (-103, 134, 29)^T$; in the Cartesian plan, the reflection point is $(-\frac{103}{29}, \frac{134}{29})$.

6. (a) The rotation matrix $\text{Rot}_{zyz}(\alpha, \beta, \gamma)$ is the concatenation of the three rotation matrix about axes z_B , y_B , and z_B , i.e., $\text{Rot}_z(\alpha) \circ \text{Rot}_y(\beta) \circ \text{Rot}_z(\gamma)$. Thus

$$\begin{aligned} \text{Rot}_{zyz}(\alpha, \beta, \gamma) &= \begin{pmatrix} \cos \alpha & -\sin \alpha & 0 \\ \sin \alpha & \cos \alpha & 0 \\ 0 & 0 & 1 \end{pmatrix} \circ \begin{pmatrix} \cos \beta & 0 & \sin \beta \\ 0 & 1 & 0 \\ -\sin \beta & 0 & \cos \beta \end{pmatrix} \circ \begin{pmatrix} \cos \gamma & -\sin \gamma & 0 \\ \sin \gamma & \cos \gamma & 0 \\ 0 & 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} \cos \alpha \cos \beta \cos \gamma - \sin \alpha \sin \gamma & -\cos \alpha \cos \beta \sin \gamma - \sin \alpha \cos \gamma & \cos \alpha \sin \beta \\ \sin \alpha \cos \beta \cos \gamma + \cos \alpha \sin \gamma & -\sin \alpha \cos \beta \sin \gamma + \cos \alpha \cos \gamma & \sin \alpha \sin \beta \\ -\sin \beta \cos \gamma & \sin \beta \sin \gamma & \cos \beta \end{pmatrix} \end{aligned}$$

Note that the three rotation angles are about the body frame instead of world frame. In space,

three rotations taken about fixed axes yield the same final orientation as the same three rotations taken in opposite order about the axes of the moving frame.

- (b) Given that

$$\text{Rot}_{zyz}(\alpha, \beta, \gamma) = \begin{pmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{pmatrix},$$

we consider the cases where $\sin \beta \neq 0$ and $\sin \beta = 0$ separately.

- i) If $\sin \beta \neq 0$, i.e., $|r_{33}| \neq 1$, we can assume that $\sin \beta > 0$. (That $\sin \beta < 0$ will just generate different Euler angles that result in the same rotation.) The function $\text{atan2}(s, c)$ gives the unique angle $\theta \in (-\pi, \pi]$ such that $\sin \theta = s$ and $\cos \theta = c$. Then

$$\begin{aligned} \beta &= \text{atan2}(\sqrt{r_{31}^2 + r_{32}^2}, r_{33}), \\ \alpha &= \text{atan2}(r_{23}/\sin \beta, r_{13}/\sin \beta), \\ \gamma &= \text{atan2}(r_{32}/\sin \beta, -r_{31}/\sin \beta). \end{aligned}$$

- ii) If $\sin \beta = 0$, i.e., $|r_{33}| = 1$, the above solution degenerates, and only the sum or the difference of α and γ may be computed. If $\beta = 0$, then the rotation matrix becomes

$$\begin{pmatrix} \cos \alpha \cos \gamma - \sin \alpha \sin \gamma & -\cos \alpha \sin \gamma - \sin \alpha \cos \gamma & 0 \\ \sin \alpha \cos \gamma + \cos \alpha \sin \gamma & -\sin \alpha \sin \gamma + \cos \alpha \cos \gamma & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} \cos(\alpha + \gamma) & -\sin(\alpha + \gamma) & 0 \\ \sin(\alpha + \gamma) & \cos(\alpha + \gamma) & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

which gives

$$\alpha + \gamma = \text{atan2}(r_{21}, r_{22}).$$

If $\beta = \pi$, then the rotation matrix becomes

$$\begin{pmatrix} -\cos \alpha \cos \gamma - \sin \alpha \sin \gamma & \cos \alpha \sin \gamma - \sin \alpha \cos \gamma & 0 \\ -\sin \alpha \cos \gamma + \cos \alpha \sin \gamma & \sin \alpha \sin \gamma + \cos \alpha \cos \gamma & 0 \\ 0 & 0 & -1 \end{pmatrix} = \begin{pmatrix} -\cos(\gamma - \alpha) & \sin(\gamma - \alpha) & 0 \\ \sin(\gamma - \alpha) & \cos(\gamma - \alpha) & 0 \\ 0 & 0 & -1 \end{pmatrix},$$

which gives

$$\gamma - \alpha = \text{atan2}(r_{21}, r_{22}).$$