# Planar Transformations

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## 1 Introduction

Geometry lies at the core of computer graphics, computer-aided design, computer vision, robotics, etc. We begin this course by describing how points and lines can be represented by Cartesian and homogeneous coordinates. We will introduce planar and spatial transformations to construct objects from 'geometric primitives', and to manipulate existing objects. Then we will study projections and look at how to render three-dimensional (3D) objects on a computer screen. This will be followed by an introduction to quaternions which constitute a very powerful tool in dealing with rotations. Near the end of this course, we will get back to geometry again, only to skim the surface of differential geometry with a study of planar and spatial curves.

In graphics applications, geometric objects are defined in terms of a number of building blocks called *graphical primitives*. These primitives may correspond to points, lines, curves, and surfaces. For example, a rectangle can be defined by its four sides (or four vertices). Each side is constructed from a line segment by applying a number of geometric operations, called transformations, which position, orientate, or scale the line primitives. Five types of transformation are particularly relevant in applications, namely, translations, scalings, reflections, rotations, and shears. They are our subjects in this first lecture.

Recall that the general equation of a line is given as

$$ax + by + c = 0, \qquad \text{where } a \neq 0 \text{ or } b \neq 0. \tag{1}$$

The equation above is the *implicit form* of the line. We normalize the coefficients and obtain

$$\frac{a}{\sqrt{a^2+b^2}}\,x + \frac{b}{\sqrt{a^2+b^2}}\,y + \frac{c}{\sqrt{a^2+b^2}} = 0. \tag{2}$$

For convenience, we introduce the unit vector $\boldsymbol{n} = (\frac{a}{\sqrt{a^2+b^2}}, \frac{b}{\sqrt{a^2+b^2}})^T$. Now, move the term involving $c$ to the right hand side of equation (2) and rewrite the remaining terms on the left hand side into a dot product:

$$\boldsymbol{n} \cdot \begin{pmatrix} x \\ y \end{pmatrix} = -\frac{c}{\sqrt{a^2+b^2}}. \tag{3}$$

For any two points $\binom{x_1}{y_1}$ and $\binom{x_2}{y_2}$ on the line, we have that

$$\boldsymbol{n} \cdot \left( \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} - \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} \right) = 0.$$

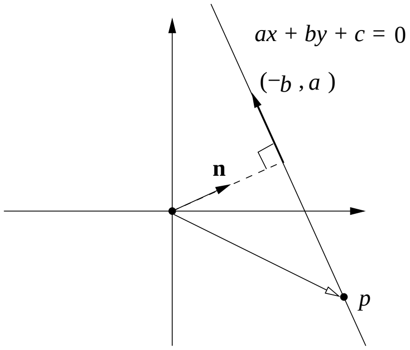

**Figure 1**: A line.

Thus the unit vector $\boldsymbol{n}$ is perpendicular to the line. Equation (3) states that the distance to the line from the origin is $\frac{|c|}{\sqrt{a^2+b^2}}$. The vector $\boldsymbol{n}$ points toward the line when $c < 0$ and away from the line when $c > 0$. We easily see that the vector $\binom{a}{b}$, just like $\boldsymbol{n}$, is perpendicular to the line while the orthogonal vector $\binom{-b}{a}$ is parallel to the line.

The line through a point $\boldsymbol{p} = \binom{p_1}{p_2}$ in the direction of the vector $\boldsymbol{v} = \binom{v_1}{v_2}$ can be defined parametrically as

$$\begin{aligned} \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} &= \boldsymbol{p} + t\boldsymbol{v} \qquad (4) \\ &= \begin{pmatrix} p_1 + tv_1 \\ p_2 + tv_2 \end{pmatrix}. \end{aligned}$$

From this parametric (or explicit) form, we immediately derive the implicit form of the line by eliminating $t$ from $x = p_1 + v_1 t$ and $y = p_2 + v_2 t$:

$$v_2 x - v_1 y + (p_2 v_1 - p_1 v_2) = 0.$$

Conversely, given the implicit form (1), we may set $\boldsymbol{v} = \binom{-b}{a}$ parallel to the line in deriving the parametric form (4). A point on the line can be chosen by setting $x = 0$ in case $b \neq 0$ (or $y = 0$ otherwise) and determining $y$ (or $x$, respectively).

Example 1. Consider two lines $a_1x + b_1y + c_1 = 0$ and $a_2x + b_2y + c_2 = 0$. Their directions are $\boldsymbol{v} = (-b_1, a_1)$ and $\boldsymbol{w} = (-b_2, a_2)$, respectively. Let $\theta$ be the angle between the two lines, more specifically, from $\boldsymbol{v}$ to $\boldsymbol{w}$. Then the identities $\boldsymbol{v} \cdot \boldsymbol{w} = \|\boldsymbol{v}\| \cdot \|\boldsymbol{w}\| \cos\theta$ and $\boldsymbol{v} \times \boldsymbol{w} = \|\boldsymbol{v}\| \cdot \|\boldsymbol{w}\| \sin\theta$ give rise to

$$\begin{aligned} \cos\theta &= \frac{a_1a_2 + b_1b_2}{\sqrt{a_1^2 + b_1^2}\sqrt{a_2^2 + b_2^2}}, \\ \sin\theta &= \frac{a_1b_2 - a_2b_1}{\sqrt{a_1^2 + b_1^2}\sqrt{a_2^2 + b_2^2}}. \end{aligned}$$

Hence

$$\tan\theta = \frac{a_1b_2 - a_2b_1}{a_1a_2 + b_1b_2}.$$

The two lines are parallel if and only if $\theta = 0$, that is, if and only if $a_1b_2 = a_2b_1$.

A *linear transformation* of the plane is a mapping $L : \mathbb{R}^2 \to \mathbb{R}^2$ from the plane to itself such that

$$\begin{pmatrix} x \\ y \end{pmatrix} \mapsto A \begin{pmatrix} x \\ y \end{pmatrix} + \boldsymbol{b}, \tag{5}$$

where

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \qquad \text{and} \qquad \boldsymbol{b} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}.$$

A linear transformation is also called an *affine mapping* or *affine transformation.*

**Lemma 1** *The transformation given by (5) maps the line $cx + dy + e = 0$, where $c \neq 0$ or $d \neq 0$, to the line*

$$(a_{22}c - a_{21}d)x + (a_{11}d - a_{12}c)y + \Big((a_{12}b_2 - a_{22}b_1)c - (a_{11}b_2 - a_{21}b_1)d + (a_{11}a_{22} - a_{12}a_{21})e\Big) = 0.$$

*If $a_{11}d - a_{12}c = 0$ and $a_{22}c - a_{21}d = 0$, then $a_{11}a_{22} - a_{12}a_{21} = 0$ and every point on the original line is mapped to the point $((b_1d - a_{12}e)/d, (b_2d - a_{22}e)/d)^T$.*

**Proof** Use the parametric form to find the image of an arbitrary point on the original line. Then convert the obtained parametric coordinates of the image into an implicit equation. □

Besides collinearity, affine transformation also preserves ratios of distances [2, p. 36], for instance, the midpoint of a line segment remains the midpoint after the transformation.

Example 2. Consider the affine mapping (5) with

$$A = \begin{pmatrix} -1 & 1 \\ 0 & 1 \end{pmatrix} \qquad \text{and} \qquad \boldsymbol{b} = \begin{pmatrix} 3 \\ 4 \end{pmatrix}.$$

The square with vertices

$$\boldsymbol{v}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad \boldsymbol{v}_2 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}, \quad \boldsymbol{v}_3 = \begin{pmatrix} -1 \\ -1 \end{pmatrix}, \quad \text{and} \quad \boldsymbol{v}_4 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

is mapped to a parallelogram with vertices

$$\boldsymbol{v}_1' = \begin{pmatrix} 3 \\ 5 \end{pmatrix}, \quad \boldsymbol{v}_2' = \begin{pmatrix} 5 \\ 5 \end{pmatrix}, \quad \boldsymbol{v}_3' = \begin{pmatrix} 3 \\ 3 \end{pmatrix}, \quad \text{and} \quad \boldsymbol{v}_4' = \begin{pmatrix} 1 \\ 3 \end{pmatrix}.$$

As shown in the figure above, the square becomes a parallelogram no longer centered at the origin. The vertices are ordered *clockwise* by index in the image (as opposed to counterclockwise in the square).

## 2 Translation

A *translation* is an affine transformation (5) with the matrix $A = I$. That is, a transformation maps every point $\boldsymbol{p}$ to a new point $\boldsymbol{p}'$ by adding a constant vector $\boldsymbol{b} = \binom{b_1}{b_2}$. It has the effect of moving the point in the direction of the $x$-axis by $b_1$ units, and in the direction of the $y$-axis by $b_2$ units. We denote the translation by $\text{Trans}(b_1, b_2)$.

The transformation that maps $\boldsymbol{p}'$ back to $\boldsymbol{p}$ is the *inverse translation* $T^{-1} = \text{Trans}(-b_1, -b_2)$.

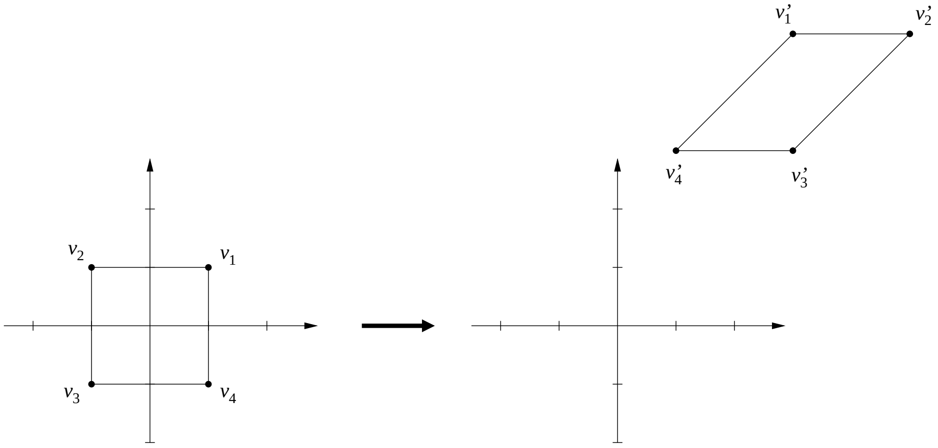

**Figure 2**: Affine transformation applied on a square of side 2.

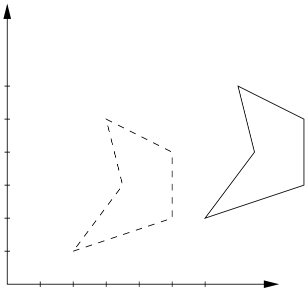

**Figure 3**: A 5-gon before and after the translation $T(4, 1)$.

# 3 Scaling

A *scaling* about the origin is an affine transformation (5) where the matrix $A = \text{diag}(s_x, s_y)$ with $s_x \neq 0$ and $s_y \neq 0$, and $\boldsymbol{b} = 0$. This transformation, denoted by $\text{Scale}(s_x, s_y)$, maps a point by multiplying its $x$ and $y$ coordinates by factors $s_x$ and $s_y$, respectively. Here $s = \sqrt{s_x^2 + s_y^2}$ is the scaling factor. The scaling is said to be an *enlargement* if $s > 1$, and a *contraction* if $s < 1$. It is said to be *uniform* if $s_x = s_y$.

Scaling can be performed by a matrix multiplication

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} s_x & 0 \\ 0 & s_y \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}.$$

We abuse the notation by letting

$$\text{Scale}(s_x, s_y) = \begin{pmatrix} s_x & 0 \\ 0 & s_y \end{pmatrix}.$$

This matrix is called the *scaling transformation matrix*.

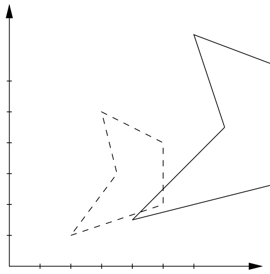

**Figure 4**: The same 5-gon in Figure 3 after scaling $S(2, 1.5)$.

# 4 Reflection

Two common effects in CAD or computer drawing packages are the horizontal or vertical 'flip' or mirror effects. A flip of an object is obtained by applying a transformation known as *reflection*. Consider a fixed line $l$ in the plane and a point $\boldsymbol{p}$, as shown in Figure 5. To determine the reflected image of $\boldsymbol{p}$, move from $\boldsymbol{p}$ toward $l$ in the direction normal to the line. Let $\boldsymbol{q}$ be the intersection of the movement with $l$. So $\boldsymbol{q}$ is the *projection* of $\boldsymbol{p}$ onto $l$ and $d = \|\boldsymbol{p} - \boldsymbol{q}\|$ gives the shortest distance from $\boldsymbol{p}$ to $l$. A continuing movement from $\boldsymbol{q}$ for another distance of $d$ will reach $\boldsymbol{p}'$, the reflection of $\boldsymbol{p}$.

It is easy to verify that the reflection $\text{Ref}_x$ in the $x$-axis is the transformation $\binom{x}{y} \mapsto \binom{x}{-y}$, and the reflection $\text{Ref}_y$ in the $y$-axis is the transformation $\binom{x}{y} \mapsto \binom{-x}{y}$. These two transformations can be denoted by matrices

$$\begin{aligned} \text{Ref}_x &= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \\ \text{Ref}_y &= \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}. \end{aligned}$$

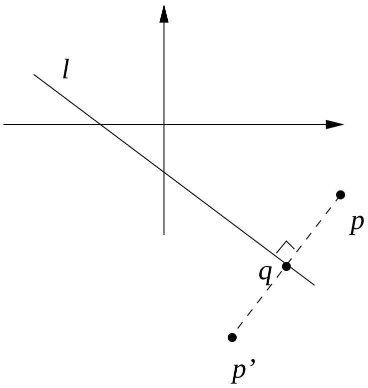

**Figure 5**: Reflection $\boldsymbol{p}'$ of a point $\boldsymbol{p}$ about a line $l$.

Reflections in arbitrary lines can also be denoted by matrices. We will easily derive such matrices after the introduction of homogeneous coordinates.

## 5 Rotation about the Origin

A *rotation* about the origin through an angle $\theta$ maps every point $\boldsymbol{p} = \binom{x}{y}$ to a point $\boldsymbol{p}' = \binom{x'}{y'}$ such that $\boldsymbol{p}$ and $\boldsymbol{p}'$ are at the same distance from the origin and the angle from the vector $\boldsymbol{p}$ to the vector $\boldsymbol{p}'$ is $\theta$. See Figure 6.

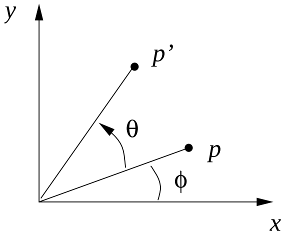

**Figure 6**: Rotation about the origin.

To determine the coordinates of the image point $\boldsymbol{p}'$, it is very convenient for us to use polar coordinates. Let $\binom{x}{y} = (r\cos\phi, r\sin\phi)^T$, where $r$ is the distance from $\boldsymbol{p}$ to the origin and $\phi$ the polar angle. Then we have

$$\begin{aligned}
x' &= r\cos(\theta+\phi) \\
&= r\cos\theta\cos\phi - r\sin\theta\sin\phi \\
&= x\cos\theta - y\sin\theta, \\
y' &= r\sin(\theta+\phi) \\
&= r\sin\theta\cos\phi + r\cos\theta\sin\phi \\
&= x\sin\theta + y\cos\theta.
\end{aligned}$$

More succinctly, the coordinates of $\boldsymbol{p}'$ can be obtained from $\boldsymbol{p}'$ through a matrix multiplication:

$$\boldsymbol{p}' = \begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}.$$

The orthogonal matrix[1]

$$\mathrm{Rot}(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

is called the *rotation matrix*. The inverse transform is the transpose $\mathrm{Rot}(\theta)^T$ which rotates vectors back through $-\theta$.

In the next figure, the 5-gon in Figure 3 has been rotated by 110 degrees.

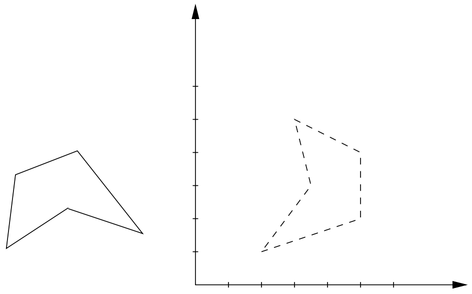

**Figure 7**: Rotation of a 5-gon about the origin by 110 degrees.

Example 3. Consider a planar 2R robot manipulator (see Figure 8) consisting of two rigid links. The first link is attached to the base by a revolute joint $J_1$ which permits the link to rotate about the joint. The second link is attached to the first link by another revolute joint $J_2$. The robot's end effector is attached to the second link. The *pose* (i.e., position and orientation) of the end effector is controlled by exerting internal torques to turn the links about the two joints

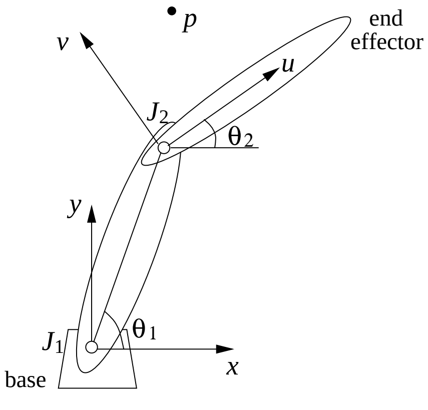

**Figure 8**: 2R robot manipulator.

We set up a world $(x, y)$-coordinate system with $J_1$ as the origin. The second link also has its own $(u, v)$-coordinate system with $J_2$ as the origin. Let $d$ be the distance between $J_1$ and $J_2$, $\theta_1$ be the angle

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[1] A square matrix $Q$ is *orthogonal* if $QQ^T = Q^TQ = I$.

between the first link and the $x$-axis, $\theta_2$ be the angle between the second link and the $x$-axis. The pose of the second link is obtained by applying a rotation $\mathrm{Rot}(\theta_2)$ followed by a translation $\mathrm{Trans}(d\cos\theta_1, d\sin\theta_1)$. Given the $(u, v)$ coordinate of a point $\boldsymbol{p}$ with respect to the second link, the $(x, y)$ coordinates of $\boldsymbol{p}$ in the world coordinate system is obtained by the transformation

$$\begin{aligned}\begin{pmatrix} x \\ y \end{pmatrix} &= \begin{pmatrix} \cos\theta_2 & -\sin\theta_2 \\ \sin\theta_2 & \cos\theta_2 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} + \begin{pmatrix} d\cos\theta_1 \\ d\sin\theta_1 \end{pmatrix} \\ &= \begin{pmatrix} u\cos\theta_2 - v\sin\theta_2 + d\cos\theta_1 \\ u\sin\theta_2 + v\cos\theta_2 + d\sin\theta_1 \end{pmatrix}.\end{aligned}$$

When the joint angles are known, the world coordinates of an object can be determined from its local coordinates with respect to the robot's end effector. The calculation is referred to as the *forward kinematics* of the robot manipulator. We can generalize the above result to a robot manipulator with $n$ revolute joints. Our ultimate aim is to express the concatenations of all rotations and translations associated with the joints with one matrix multiplication. This will be possible with the assistance of homogeneous coordinates.

# 6 Shear

Let a fixed direction be represented by the unit vector $\boldsymbol{v} = \binom{v_1}{v_2}$. A *shear* about the origin of factor $r$ in the direction $\boldsymbol{v}$ maps a point $\boldsymbol{p}$ to the point $\boldsymbol{p}' = \boldsymbol{p} + dr\boldsymbol{v}$, where $d$ is the (signed) distance from the origin to the line through $\boldsymbol{p}$ in the direction $\boldsymbol{v}$.

Suppose $\boldsymbol{p} = \binom{x}{y}$. The unit vector normal to the line is $\boldsymbol{n} = \binom{-v_2}{v_1}$. Note that we choose the normal vector such that $\boldsymbol{v} \times \boldsymbol{n} = 1$. Therefore the distance $d$ is given as

$$d = \boldsymbol{p} \cdot \boldsymbol{n} = yv_1 - xv_2.$$

The shear transformation then maps $\boldsymbol{p}$ to

$$\boldsymbol{p}' = \boldsymbol{p} + dr\boldsymbol{v} = \begin{pmatrix} x + r(v_1 y - v_2 x)v_1 \\ y + r(v_1 y - v_2 x)v_2 \end{pmatrix}.$$

Thus the shear transformation matrix is

$$\mathrm{Shear}(\boldsymbol{v}, r) = \begin{pmatrix} 1 - rv_1v_2 & rv_1^2 \\ -rv_2^2 & 1 + rv_1v_2 \end{pmatrix}.$$

In particular, a shear along the $x$-axis has $\boldsymbol{v} = \binom{1}{0}$ and thus

$$\mathrm{Shear}\left(\binom{1}{0}, r\right) = \begin{pmatrix} 1 & r \\ 0 & 1 \end{pmatrix}$$

.

Example 4. The shear in the direction $\boldsymbol{v} = (\frac{2}{\sqrt{5}}, \frac{1}{\sqrt{5}})$ with a factor $r = \frac{3}{2}$ has transformation matrix

$$\begin{aligned}\mathrm{Shear}\left((\tfrac{2}{\sqrt{5}}, \tfrac{1}{\sqrt{5}})^T, \tfrac{3}{2}\right) &= \begin{pmatrix} 1 - \frac{3}{2}\frac{2}{\sqrt{5}}\frac{1}{\sqrt{5}} & \frac{3}{2}\left(\frac{2}{\sqrt{5}}\right)^2 \\ -\frac{3}{2}\left(\frac{1}{\sqrt{5}}\right)^2 & 1 + \frac{3}{2}\frac{2}{\sqrt{5}}\frac{1}{\sqrt{5}} \end{pmatrix} \\ &= \begin{pmatrix} \frac{2}{5} & \frac{6}{5} \\ -\frac{3}{10} & \frac{8}{5} \end{pmatrix}.\end{aligned}$$

Applying the shear to the 5-gon in Example 1, we have

$$\begin{pmatrix} \frac{2}{5} & \frac{6}{5} \\ -\frac{3}{10} & \frac{8}{5} \end{pmatrix} \begin{pmatrix} 2 & 5 & 5 & 3 & \frac{7}{2} \\ 1 & 2 & 4 & 5 & 3 \end{pmatrix} = \begin{pmatrix} 2 & 4.4 & 6.8 & 7.2 & 5 \\ 1 & 1.7 & 4.9 & 7.1 & 3.75 \end{pmatrix}.$$

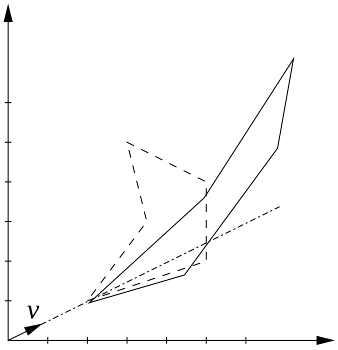

**Figure 9**: Shearing in $\boldsymbol{v} = (\frac{2}{\sqrt{5}}, \frac{1}{\sqrt{5}})$ by a factor $r = \frac{3}{2}$. The portion of the 5-gon to the left of $\boldsymbol{v}$ is extended along the direction $\boldsymbol{v}$ while the portion to the right of the vector is pulled back in the direction $-\boldsymbol{v}$.

## References

[1] D. Marsh. *Applied Geometry for Computer Graphics and CAD*. Springer-Verlag, 1999.

[2] E. W. Weisstein. *CRC Concise Enclyclopedia of Mathematics*, 2nd ed. Chapman & Hall/CRC, 2003.