Transformations in Homogeneous Coordinates

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1 Homogeneous Transformations

A *projective transformation* of the projective plane is a mapping $L : \mathbb{P}^2 \to \mathbb{P}^2$ defined as

$$\begin{pmatrix} u \\ v \\ w \end{pmatrix} \mapsto \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & k \end{pmatrix} \begin{pmatrix} u \\ v \\ w \end{pmatrix} = \begin{pmatrix} au + bv + cw \\ du + ev + fw \\ gu + hv + kw \end{pmatrix}, \tag{1}$$

where $a, b, c, d, e, f, g, h, k \in \mathbb{R}$. The 3×3 matrix representing the mapping L of the projective plane is called a *homogeneous transformation matrix*. When $g = h = 0$ and $k \neq 0$, the mapping L is an *affine transformation* introduced in the previous lecture. Affine transformations correspond to transformations of the Cartesian plane.

Note that homogeneous coordinates (ru, rv, rw) under the mapping (1) has image

$$\begin{pmatrix} aru + brv + crw \\ dru + erv + frw \\ gru + hrv + krw \end{pmatrix}.$$

A division by r gives the image of (u, v, w). Thus $L(ru, rv, rw)$ and $L(u, v, w)$ are equivalent and correspond to the same point in homogeneous coordinates. Therefore the definition of a transformation does not depend on the choice of homogeneous coordinates for a given point.

A *transformation* of the projective space is a mapping $M : \mathbb{P}^3 \to \mathbb{P}^3$ given by

$$\begin{pmatrix} s \\ u \\ v \\ w \end{pmatrix} \mapsto \begin{pmatrix} m_{11} & m_{12} & m_{13} & m_{14} \\ m_{21} & m_{22} & m_{23} & m_{24} \\ m_{31} & m_{32} & m_{33} & m_{34} \\ m_{41} & m_{42} & m_{43} & m_{44} \end{pmatrix} \begin{pmatrix} s \\ u \\ v \\ w \end{pmatrix}.$$

The 4×4 matrix (m_{ij}) is called the *homogeneous transformation matrix* of M. If the matrix is non-singular, then M is called a *non-singular* transformation. If $m_{41} = m_{42} = m_{43} = 0$ and $m_{44} \neq 0$, then M is said to be an *affine transformation*. Affine transformations correspond to translations, scalings, rotations, reflections, etc. of the three-dimensional space.

2 Translation and Scaling

We first describe the homogeneous transformation matrices for translations and scalings, in the plane and the space. Let us start with translation:

$$\text{Trans}(h,k) = \begin{pmatrix} 1 & 0 & h \\ 0 & 1 & k \\ 0 & 0 & 1 \end{pmatrix}.$$

Then

$$\begin{pmatrix} 1 & 0 & h \\ 0 & 1 & k \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ 1 \end{pmatrix} = \begin{pmatrix} x+h \\ y+k \\ 1 \end{pmatrix},$$

which verifies that the point (x,y) is translated to $(x+h, y+k)$.

A translation by a, b, c in the x-, y-, and z-directions, respectively, has the transformation matrix:

$$\text{Trans}(a,b,c) = \begin{pmatrix} 1 & 0 & 0 & a \\ 0 & 1 & 0 & b \\ 0 & 0 & 1 & c \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

The point $\boldsymbol{p} = (x,y,z,1)$ is translated to the point

$$\begin{aligned} \boldsymbol{p}' &= \text{Trans}(a,b,c)\,\boldsymbol{p} \\ &= \begin{pmatrix} 1 & 0 & 0 & a \\ 0 & 1 & 0 & b \\ 0 & 0 & 1 & c \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ 1 \end{pmatrix} \\ &= \begin{pmatrix} x+a \\ y+b \\ z+c \\ 1 \end{pmatrix}. \end{aligned}$$

Accordingly, the point (x,y,z) in the Cartesian space is translated to $(x+a, y+b, z+c)$.

The homogeneous scaling matrix is

$$\text{Scale}(s_x, s_y) = \begin{pmatrix} s_x & 0 & 0 \\ 0 & s_y & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Then

$$\begin{pmatrix} s_x & 0 & 0 \\ 0 & s_y & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ 1 \end{pmatrix} = \begin{pmatrix} s_x x \\ s_y y \\ 1 \end{pmatrix}.$$

Example 1. The unit square with vertices $(1,1)$, $(2,1)$, $(2,2)$, and $(1,2)$ is scaled about the origin by factors of 4 and 2 in the x- and y- directions, respectively. We have

$$\begin{pmatrix} 4 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 & 2 & 1 \\ 1 & 1 & 2 & 2 \\ 1 & 1 & 1 & 1 \end{pmatrix} = \begin{pmatrix} 4 & 8 & 8 & 4 \\ 2 & 2 & 4 & 4 \\ 1 & 1 & 1 & 1 \end{pmatrix}.$$

So the image is a square with vertices $(4,2)$, $(8,2)$, $(8,4)$, and $(4,4)$.

A scaling about the origin by factors s_x/s_w, s_y/s_w, and s_z/s_w in the x-, y-, and z-directions, respectively, has the transformation matrix (often, s_w is chosen to be 1):

$$\text{Scale}(s_x, s_y, s_z, s_w) = \begin{pmatrix} s_x & 0 & 0 & 0 \\ 0 & s_y & 0 & 0 \\ 0 & 0 & s_z & 0 \\ 0 & 0 & 0 & s_w \end{pmatrix}.$$

Similar to the cases of translation and scaling, the transformation matrix for a planar rotation about the origin through an angle θ is

$$\text{Rot}(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

3 Rotation about a Point

A rotation through an angle θ about a point (a,b) is obtained by performing a translation which maps (a,b) to the origin, followed by a rotation through an angle θ about the origin, and followed by a translation which maps the origin to (a,b). The rotation matrix is

$$\begin{aligned} \text{Rot}_{(a,b)}(\theta) &= \text{Trans}(a,b)\,\text{Rot}(\theta)\,\text{Trans}(-a,-b) \\ &= \begin{pmatrix} 1 & 0 & a \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & -a \\ 0 & 1 & -b \\ 0 & 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} \cos\theta & -\sin\theta & -a\cos\theta + b\sin\theta + a \\ \sin\theta & \cos\theta & -a\sin\theta - b\cos\theta + b \\ 0 & 0 & 1 \end{pmatrix}. \end{aligned} \tag{2}$$

Example 2. A square has vertices $\boldsymbol{p}_1 = (1,1)$, $\boldsymbol{p}_2 = (2,1)$, $\boldsymbol{p}_3 = (2,2)$, and $\boldsymbol{p}_4(1,2)$. Determine the new vertices of the square after a rotation about $\boldsymbol{p}_2$ through an angle of $\pi/4$. The transformation matrix is

$$\text{Rot}_{(2,1)}(\pi/4) = \begin{pmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} & 2 - \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & -\frac{3\sqrt{2}}{2} + 1 \\ 0 & 0 & 1 \end{pmatrix}.$$

Apply the transformation above to the homogeneous coordinates of the vertices:

$$\begin{aligned} \begin{pmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} & 2 - \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & -\frac{3\sqrt{2}}{2} + 1 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 & 2 & 1 \\ 1 & 1 & 2 & 2 \\ 1 & 1 & 1 & 1 \end{pmatrix} &= \begin{pmatrix} 2 - \frac{\sqrt{2}}{2} & 2 & 2 - \frac{\sqrt{2}}{2} & 2 - \sqrt{2} \\ 1 - \frac{\sqrt{2}}{2} & 1 & 1 + \frac{\sqrt{2}}{2} & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 1.2929 & 2 & 1.2929 & 0.5858 \\ 0.2929 & 1 & 1.7071 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix}. \end{aligned}$$

Thus the new vertices are $\boldsymbol{p}_1' = (1.2929, 0.2929)$, $\boldsymbol{p}_2' = (2,1)$, $\boldsymbol{p}_3' = (1.2929, 1.7071)$, and $\boldsymbol{p}_4' = (0.5858, 1.0)$, as illustrated in Figure 1.

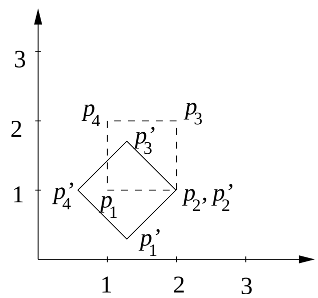


Figure 1: Rotation about p_2.

4 Inverse Transformations

The *identity transformation* I is the transformation that leaves all points of the plane unchanged. More precisely, I is the transformation for which $I \circ L = L \circ I = L$, for any planar transformation L. The transformation matrix of the identity transformation in homogeneous coordinates is the 3×3 identity matrix I_3.

The *inverse* of a transformation L, denoted L^{-1}, maps images of L back to the original points. More precisely, the inverse L^{-1} satisfies that $L^{-1} \circ L = L \circ L^{-1} = I$.

Lemma 1 *Let T be the matrix of the homogeneous transformation L. If the inverse transformation L^{-1} exists, then T^{-1} exists and is the transformation matrix of L^{-1}. Conversely, if T^{-1} exists, then the transformation represented by T^{-1} is the inverse transformation of L.*

Proof Suppose L has an inverse L^{-1} with transformation matrix R. The concatenations $L \circ L^{-1}$ and $L^{-1} \circ L$ must be identity transformations. Accordingly, the transformation matrices TR and RT are equal to I_3. Thus R is the inverse of matrix T, that is, $R = T^{-1}$.

Conversely, suppose the matrix T has an inverse T^{-1}, which defines a transformation R. Since $T^{-1}T = TT^{-1} = I_3$, it follows that $R \circ L$ and $L \circ R$ are the identity transformation. By definition, R is the inverse transformation. □

We easily obtain the inverses of translation, rotation, and scaling:

$$\begin{aligned} \text{Trans}(h,k)^{-1} &= \text{Trans}(-h,-k), \\ \text{Rot}(\theta)^{-1} &= \text{Rot}(-\theta), \\ \text{Scale}(s_1,s_2)^{-1} &= \text{Scale}(1/s_1, 1/s_2). \end{aligned}$$

A transformation $L : \mathbb{P}^2 \to \mathbb{P}^2$ with an inverse L^{-1} is called a *non-singular transformation.* Lemma 1 implies that a transformation is non-singular if and only if its transformation matrix is non-singular. Non-singular matrices A and B satisfies $(AB)^{-1} = B^{-1}A^{-1}$.

In homogeneous coordinates, the concatenation of transformations T_1 and T_2, denoted $T_2 \circ T_1$, can be carried out with matrix multiplications alone. For example, a rotation $\text{Rot}(\theta)$ about the origin followed by a translation $\text{Trans}(h,k)$ followed by a scaling $\text{Scale}(s_x,s_y)$ has the homogeneous transformation matrix

$$\text{Scale}(s_x,s_y) \circ \text{Trans}(h,k) \circ \text{Rot}(\theta) = \begin{pmatrix} s_x & 0 & 0 \\ 0 & s_y & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & h \\ 0 & 1 & k \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} s_x \cos\theta & -s_x \sin\theta & s_x h \\ s_y \sin\theta & s_y \cos\theta & s_y k \\ 0 & 0 & 1 \end{pmatrix}$$

Example 3. Determine the transformation matrix of the inverse of the concatenation of transformations $\mathrm{Trans}(-2,5) \circ \mathrm{Rot}(-\pi/3)$. The transformation matrix L^{-1} of the inverse is the inverse of the corresponding matrix product:

$$\begin{aligned} L^{-1} &= \Big(\mathrm{Trans}(-2,5)\,\mathrm{Rot}(-\pi/3)\Big)^{-1} \\ &= \mathrm{Rot}(-\pi/3)^{-1}\,\mathrm{Trans}(-2,5)^{-1} \\ &= \mathrm{Rot}(\pi/3)\,\mathrm{Trans}(2,-5) \\ &= \begin{pmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} & 1+\frac{5\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} & -\frac{5}{2}+\sqrt{3} \\ 0 & 0 & 1 \end{pmatrix}. \end{aligned}$$

5 Reflection in an Arbitrary Line

How to determine the transformation matrix for reflection in an arbitrary line $ax + by + c = 0$? If $c = 0$ and either $a = 0$ or $b = 0$, then it is the reflection in either x- or y-axis that we considered before. The matrix is

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{3}$$

In general, the reflection is obtained by transforming the line to one of the axes, reflecting in that axis, and then taking the inverse of the first transformation. Suppose $b \neq 0$. More specifically, the reflection is accomplished in the following five steps:

1. The line intersects the y-axis in the point $(0, -c/b)$.
2. Make a translation that maps $(0, -c/b)$ to the origin.
3. The slope of the line is $\tan\theta = -a/b$, where the angle θ made by the line with the x-axis remains the same after the translation. Rotate the line about the origin through an angle $-\theta$. This maps the line to the x-axis.
4. Apply a reflection in the x-axis.
5. Rotate about the origin by $-\theta$ and then translate by $(0, -c/b)$.

The concatenation of the above transformation is

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -c/b \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\times \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & c/b \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} \cos^2\theta - \sin^2\theta & 2\sin\theta\cos\theta & \frac{2c}{b}\sin\theta\cos\theta \\ 2\sin\theta\cos\theta & \sin^2\theta - \cos^2\theta & -\frac{2c}{b}\cos^2\theta \\ 0 & 0 & 1 \end{pmatrix}.$$

Since $\tan\theta = -a/b$, it follows that $\cos^2\theta = 1/(1+\tan^2\theta) = b^2/(a^2+b^2)$ and $\sin^2\theta = 1-\cos^2\theta = a^2/(a^2+b^2)$. So $\sin\theta\cos\theta = \tan\theta\cos^2\theta = -ab/(a^2+b^2)$. Substituting these expressions into (4) yields

$$\begin{pmatrix} \frac{b^2-a^2}{a^2+b^2} & -\frac{2ab}{a^2+b^2} & -\frac{2ac}{a^2+b^2} \\ -\frac{2ab}{a^2+b^2} & \frac{a^2-b^2}{a^2+b^2} & -\frac{2bc}{a^2+b^2} \\ 0 & 0 & 1 \end{pmatrix}.$$

In homogeneous coordinates, multiplication by a factor does not change the point. So the above matrix can be scaled by a factor a^2+b^2 to remove all the denominators in the entries, yielding the reflection matrix

$$\mathrm{Ref}_{(a,b,c)} = \begin{pmatrix} b^2-a^2 & -2ab & -2ac \\ -2ab & a^2-b^2 & -2bc \\ 0 & 0 & a^2+b^2 \end{pmatrix}.$$

Note that the above matrix agrees with (3) in the cases of reflection in x- and y-axes. Thus we have removed the assumption $b \neq 0$ made for deriving the reflection matrix.

6 Application 1 — Instancing

A geometric object is created by defining its components. For example, the front of a house in Figure 2 consists of rectangles, which form the walls, windows, and door of the house. The rectangles

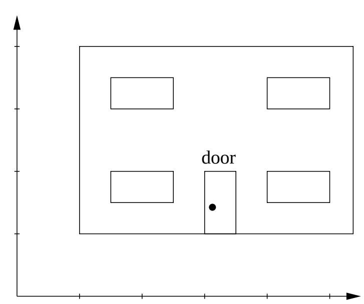


Figure 2: Front of a house obtained from instances of square and point.

are scaled from a square, which is an example of a *picture element*. For convenience, picture elements are defined in their own local coordinate systems, and are constructed from *graphical primitives* which are the basic building blocks. Picture elements are defined once but may be used many times in the construction of objects.

For example, a square with vertices $(0,0)$, $(1,0)$, $(1,1)$, and $(0,1)$ can be obtained using the graphical primitive `Line` for the line segment joining the points $(0,0)$ and $(1,0)$ through rotations and translations. A transformed copy of a graphical primitive or picture element is referred to as an instance. The aforementioned square, denoted `Square`, is defined by four instances of the line.

The completed 'real' object is defined in world coordinates by applying a transformation to each picture element. The house in the figure is defined by six instances of the picture element `Square`, and one instance of the primitive `Point` (for the door handle). In particular, the front door is obtained from `Square` by applying a scaling of 0.5 unit in the x-direction, followed by a translation of 3 units in the x-direction and 1 unit in the y-direction. In homogeneous coordinates, the transformation matrix is given by

$$
\begin{aligned}
\mathrm{Trans}(3,1) \circ \mathrm{Scale}(0.5,1) &= \begin{pmatrix} 1 & 0 & 3 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0.5 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \\
&= \begin{pmatrix} 0.5 & 0 & 3 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}.
\end{aligned}
$$

The vertices of the door are obtained by applying the above transformation matrix to the vertices of the `Square` primitive, giving

$$
\begin{pmatrix} 0.5 & 0 & 3 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix} = \begin{pmatrix} 3 & 3.5 & 3.5 & 3 \\ 1 & 1 & 2 & 2 \\ 1 & 1 & 1 & 1 \end{pmatrix}.
$$

So the vertices of the door in world coordinates are $(3,1)$, $(3.5,1)$, $(3.5,2)$, and $(3,2)$.

7 Rotation in Space

Like a rotation in the plane, a rotation in the space takes about a line referred to as its *rotation axis*. Any rotation can be decomposed into three primary rotations about the x-, and y-, and z-axes:

$$
\begin{aligned}
\mathrm{Rot}_x(\theta_x) &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta_x & -\sin\theta_x & 0 \\ 0 & \sin\theta_x & \cos\theta_x & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \\
\mathrm{Rot}_y(\theta_y) &= \begin{pmatrix} \cos\theta_y & 0 & \sin\theta_y & 0 \\ 0 & 1 & 0 & 0 \\ -\sin\theta_y & 0 & \cos\theta_y & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \\
\mathrm{Rot}_z(\theta_z) &= \begin{pmatrix} \cos\theta_z & -\sin\theta_z & 0 & 0 \\ \sin\theta_z & \cos\theta_z & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.
\end{aligned}
$$

Figure 3(a) shows the direction in which the primary rotations take when the rotation angle is positive. Figure 3(b) is a mnemonic that helps to remember the directions. For instance, the positive sense of a rotation about the y-axis has the effect of moving points on the z-axis toward the x-axis.

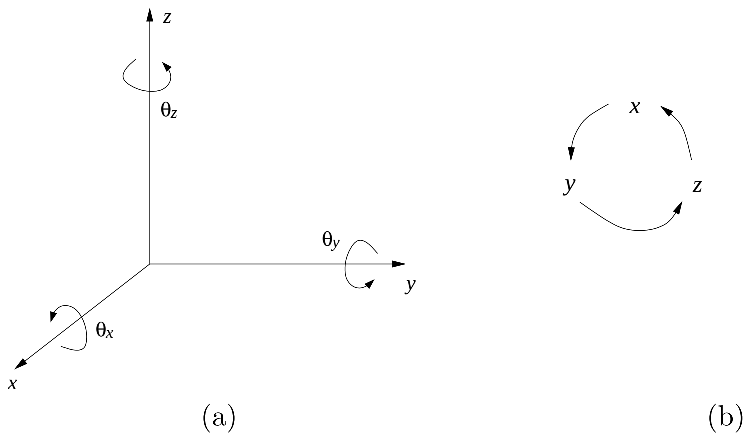


Figure 3: Rotations about the coordinate axes.

EXAMPLE 4. A rotation through an angle $\pi/6$ about the y-axis followed by a translation by 1, -1, 2 respectively along the x-, y-, and z-axes has the transformation matrix

$$
\begin{aligned}
\mathrm{Trans}(1,-1,2)\,\mathrm{Rot}_y(\pi/6) &= \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \frac{\sqrt{3}}{2} & 0 & \frac{1}{2} & 0 \\ 0 & 1 & 0 & 0 \\ -\frac{1}{2} & 0 & \frac{\sqrt{3}}{2} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \\
&= \begin{pmatrix} \frac{\sqrt{3}}{2} & 0 & \frac{1}{2} & 1 \\ 0 & 1 & 0 & -1 \\ -\frac{1}{2} & 0 & \frac{\sqrt{3}}{2} & 2 \\ 0 & 0 & 0 & 1 \end{pmatrix}.
\end{aligned}
$$

7.1 Rotation about an Arbitrary Line

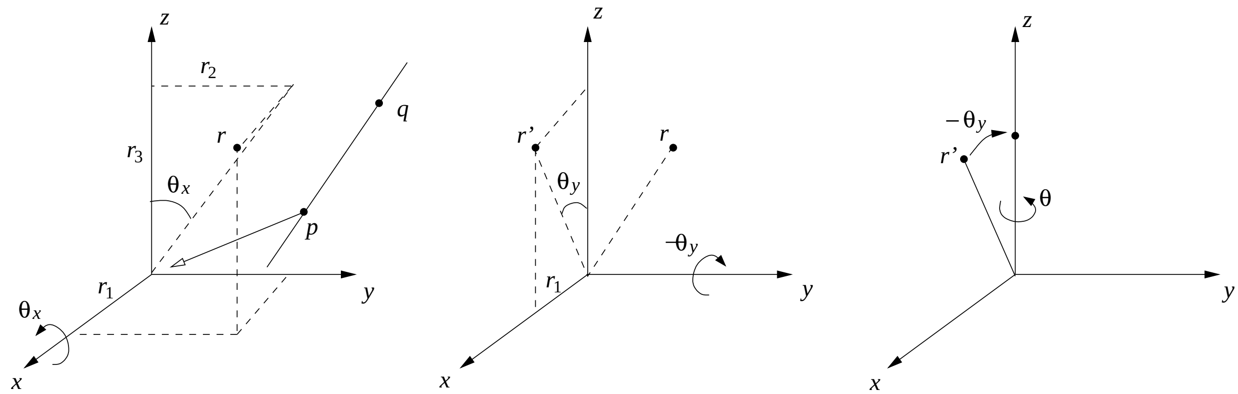
When the rotation axis is an arbitrary line, we obtain the transformation matrix as follows. Firstly, transform the rotation axis to one of the coordinate axes. Secondly, perform a rotation of the required angle θ about the coordinate axis. Finally, transform the coordinate axis back to the original rotation axis. More specifically, let the rotation axis be the line ℓ through the points $\boldsymbol{p} = (p_1, p_2, p_3)$ and $\boldsymbol{q} = (q_1, q_2, q_3)$. Denote $\boldsymbol{r} = \boldsymbol{q} - \boldsymbol{p} = (r_1, r_2, r_3)$. Then we perform the following steps:

1. Translate the point $\boldsymbol{p}$ by $(-p_1, -p_2, -p_3)$ to the origin O and the rotation axis to the line $\overline{O\boldsymbol{r}}$ through O and the point $\boldsymbol{r}$.

2. Rotate the vector $\boldsymbol{r}$ about the x-axis until it lies in the xz-plane. This is shown in Figure 4(a). Suppose the line $\overline{O\boldsymbol{r}}$ makes an angle θ_x with the xz-plane. If $r_2 = r_3 = 0$, then the line is aligned with the x-axis and $\theta_x = 0$. Otherwise, we have

$$\sin\theta_x = \frac{r_2}{\sqrt{r_2^2 + r_3^2}}, \tag{4}$$

$$\cos\theta_x = \frac{r_3}{\sqrt{r_2^2 + r_3^2}}. \tag{5}$$



(a) (b) (c)

Figure 4: Rotation about an arbitrary axis performed by transforming the axis to the z-axis, applying the rotation, and transforming back to the original axis.

The desired rotation $\text{Rot}_x(\theta_x)$ maps $\boldsymbol{r}$ to the point $\boldsymbol{r}' = (r_1, 0, \sqrt{r_2^2 + r_3^2})$ shown in Figure 4(b).

3. Rotate the vector $\boldsymbol{r}'$ about the y-axis to align it with the z-axis. This step is shown in Figure 4(b). The required angle is found to be $-\theta_y$ where

$$\sin\theta_y = \frac{r_1}{\sqrt{r_1^2 + r_2^2 + r_3^2}}, \tag{6}$$

$$\cos\theta_y = \sqrt{\frac{r_2^2 + r_3^2}{r_1^2 + r_2^2 + r_3^2}}. \tag{7}$$

4. Apply a rotation through an angle θ about the z-axis, as shown in Figure 4(c).

5. Apply the inverses of the transformations in steps 1–3 in reverse order.

Thus the general rotation through an angle θ about the line through two points $\boldsymbol{p} = (p_1, p_2, p_3)$ and $\boldsymbol{q} = (q_1, q_2, q_3)$ has the transformation matrix

$$\text{Trans}(p_1, p_2, p_3)\,\text{Rot}_x(-\theta_x)\,\text{Rot}_y(\theta_y)\text{Rot}_z(\theta)\,\text{Rot}_y(-\theta_y)\,\text{Rot}_x(\theta_x)\,\text{Trans}(-p_1, -p_2, -p_3), \tag{8}$$

where $\sin\theta_x$, $\cos\theta_x$, $\sin\theta_y$, and $\cos\theta_y$ are given in (4)–(7) with $(r_1, r_2, r_3) = \boldsymbol{q} - \boldsymbol{p}$.

Example 5. Compute the transformation matrix of the rotation through an angle θ about the line through the points $\boldsymbol{p} = (2, 1, 5)$ and $\boldsymbol{q} = (4, 7, 2)$. We have

$$\boldsymbol{r} = \boldsymbol{q} - \boldsymbol{p} = (2, 6, -3).$$

So $\sqrt{r_2^2 + r_3^2} = 3\sqrt{5}$, and $\sin\theta_x = \frac{2}{\sqrt{5}}$, $\cos\theta_x = -\frac{1}{\sqrt{5}}$, $\sin\theta_y = \frac{2}{7}$, and $\cos\theta_y = \frac{3}{7}\sqrt{5}$. The rotation matrix is

$$\begin{pmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -\frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} & 0 \\ 0 & -\frac{2}{\sqrt{5}} & -\frac{1}{\sqrt{5}} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \frac{3}{7\sqrt{5}} & 0 & \frac{2}{7} & 0 \\ 0 & 1 & 0 & 0 \\ -\frac{2}{7} & 0 & \frac{3}{7\sqrt{5}} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \cos\theta & -\sin\theta & 0 & 0 \\ \sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} \frac{3}{7\sqrt{5}} & 0 & -\frac{2}{7} & 0 \\ 0 & 1 & 0 & 0 \\ \frac{2}{7} & 0 & \frac{3}{7\sqrt{5}} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -\frac{1}{\sqrt{5}} & -\frac{2}{\sqrt{5}} & 0 \\ 0 & \frac{2}{\sqrt{5}} & -\frac{1}{\sqrt{5}} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -5 \\ 0 & 0 & 0 & 1 \end{pmatrix} =$$

$$\begin{pmatrix} \frac{45}{49}\cos\theta + \frac{4}{49} & -\frac{12}{49}\cos\theta + \frac{3}{7}\sin\theta + \frac{12}{49} & \frac{6}{49}\cos\theta + \frac{6}{7}\sin\theta - \frac{6}{49} & -\frac{108}{49}\cos\theta - \frac{33}{7}\sin\theta + \frac{108}{49} \\ -\frac{12}{49}\cos\theta - \frac{3}{7}\sin\theta + \frac{12}{49} & \frac{13}{49}\cos\theta + \frac{36}{49} & \frac{18}{49}\cos\theta - \frac{2}{7}\sin\theta - \frac{18}{49} & -\frac{79}{49}\cos\theta + \frac{16}{7}\sin\theta + \frac{79}{49} \\ \frac{6}{49}\cos\theta - \frac{6}{7}\sin\theta - \frac{6}{49} & \frac{18}{49}\cos\theta + \frac{2}{7}\sin\theta - \frac{18}{49} & \frac{40}{49}\cos\theta + \frac{9}{49} & -\frac{230}{49}\cos\theta + \frac{10}{7}\sin\theta + \frac{230}{49} \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

When the rotation axis ℓ is through the origin, we have a clean closed form for the matrix. Let $K = (k_x, k_y, k_z)$ be the unit vector along the rotation axis, and θ be the angle of rotation about this vector. Then the 3×3 rotation matrix is [2, pp. 51–52]

$$\mathrm{Rot}_K(\theta) = \begin{pmatrix} k_x k_x(1-\cos\theta) + \cos\theta & k_x k_y(1-\cos\theta) - k_z\sin\theta & k_x k_z(1-\cos\theta) + k_y\sin\theta \\ k_x k_y(1-\cos\theta) + k_z\sin\theta & k_y k_y(1-\cos\theta) + \cos\theta & k_y k_z(1-\cos\theta) - k_x\sin\theta \\ k_x k_z(1-\cos\theta) - k_y\sin\theta & k_y k_z(1-\cos\theta) + k_x\sin\theta & k_z k_z(1-\cos\theta) + \cos\theta \end{pmatrix}.$$

The homogeneous rotation matrix is given as

$$\begin{pmatrix} \mathrm{Rot}_K(\theta) & 0 \\ 0 & 1 \end{pmatrix}.$$

8 Reflection in an Arbitrary Plane

Reflection in a plane $ax + by + cz + d = 0$ is obtained by transforming the reflection plane to one of the xy-, xz-, or yz-planes, reflecting in that plane, and finally transforming the plane back to the reflection plane. More specifically, the transformation is obtained in the following steps.

1. Choose a point $\boldsymbol{p} = (p_1, p_2, p_3)$ on the plane. Translate this point to the origin so that the reflection plane becomes $ax + by + cz = 0$. Denote $\boldsymbol{r} = (a, b, c)$.

2. Then following steps 2–3 of the method of general rotation in Section 7.1 there are two angles θ_x and θ_y such that the composition of rotations $\mathrm{Rot}_y(-\theta_y) \circ \mathrm{Rot}_x(\theta_x)$ aligns the vector $\boldsymbol{r}$ with the z-axis, and maps the translated reflection plane to the xy-plane. We have $\theta_x = 0$ if $r_2 = r_3 = 0$; otherwise, $\sin\theta_x$ and $\cos\theta_x$ are defined in (4) and (5), respectively. Meanwhile, $\cos\theta_y$ and $\sin\theta_y$ are given by (6) and (7).

3. Apply the reflection in the xy-plane.

4. Apply the inverse of the transformations in steps 1–2 in reverse order.

The general reflection matrix is thus

$$\mathrm{Trans}(p_1, p_2, p_3)\,\mathrm{Rot}_x(-\theta_x)\,\mathrm{Rot}_y(\theta_y)\,\mathrm{Ref}_{xy}\,\mathrm{Rot}_y(-\theta_y)\,\mathrm{Rot}_x(\theta_x)\,\mathrm{Trans}(-p_1, -p_2, -p_3). \qquad (9)$$

We can easily verify the above reflection matrix in the special cases where the plane is parallel to the yz-plane, xy-plane, or xz-plane. In the first case, the matrix (9) reduces to

$$\mathrm{Trans}(p_1, p_2, p_3)\,\mathrm{Ref}_{yz}\,\mathrm{Trans}(-p_1, -p_2, -p_3).$$

EXAMPLE 6. Let us determine the transformation matrix for a reflection in the plane $2x - y + 2z - 2 = 0$. Pick a point, say, $(1, 0, 0)$, in the plane and translate it to the origin. The translated plane is $2x - y + 2z = 0$ which has a normal $(2, -1, 2)$. Next, we determine that

$$\begin{aligned}
\sin\theta_x &= -\frac{1}{\sqrt{5}}, \\
\cos\theta_x &= \frac{2}{\sqrt{5}}, \\
\sin(-\theta_y) &= -\sin\theta_y = -\frac{2}{3}. \\
\cos(-\theta_y) &= \cos\theta_y = \frac{\sqrt{5}}{3}.
\end{aligned}$$

The reflection matrix (9) becomes

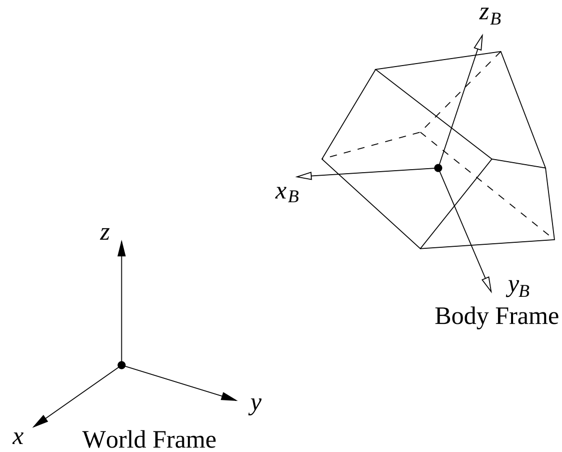
$$\begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}
\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{2}{\sqrt{5}} & -\frac{1}{\sqrt{5}} & 0 \\ 0 & \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}
\begin{pmatrix} \frac{\sqrt{5}}{3} & 0 & \frac{2}{3} & 0 \\ 0 & 1 & 0 & 0 \\ -\frac{2}{3} & 0 & \frac{\sqrt{5}}{3} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}
\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$
$$\begin{pmatrix} \frac{\sqrt{5}}{3} & 0 & -\frac{2}{3} & 0 \\ 0 & 1 & 0 & 0 \\ \frac{2}{3} & 0 & \frac{\sqrt{5}}{3} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}
\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} & 0 \\ 0 & -\frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}
\begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$
$$= \frac{1}{9}\begin{pmatrix} 1 & 4 & -8 & 8 \\ 4 & 7 & 4 & -4 \\ -8 & 4 & 1 & 8 \\ 0 & 0 & 0 & 9 \end{pmatrix}$$

We can simply remove the multiplier $\frac{1}{9}$ in front of the matrix above since homogeneous coordinates are used.

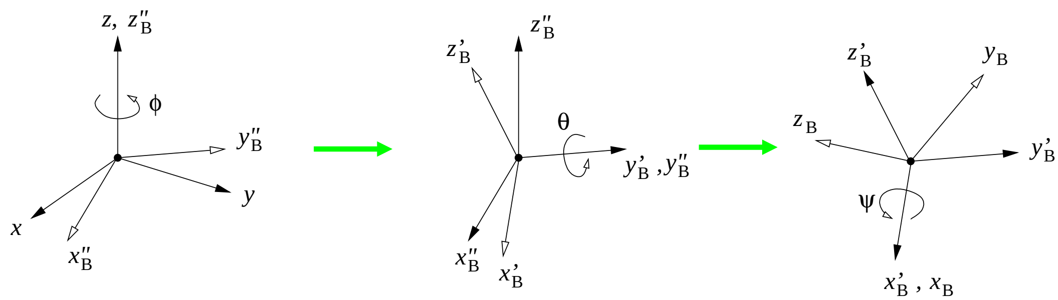
9 Application 2 — Orientation of a Rigid Body

A rigid body (such as a spaceship) has a coordinate frame attached to itself and located often at the center of mass. This frame is referred to as the *body frame* or *local frame*. The position, orientation, and motion of the body can be described using the body frame relative to a fixed reference frame, called the *world frame*. See the next figure.

The rigid body in the space has six degrees of freedom. The position of the body can be described by the x, y, z coordinates of its center of mass (i.e., the origin of the local frame) in the world frame. The orientation of the body can be described by three angles ϕ, θ, ψ which together determine the rotation of the body frame from the world frame. These three angles are called *Euler angles*. Each rotation is performed about an axis of the (moving) body frame (instead of some axis of a (fixed) world frame as described in Section 7.1).



There are several conventions for Euler angles, depending on the axes about which the rotations are carried out. Here we introduce the Z-Y-X Euler angles. The body frame starts in the same orientation as the world frame. To achieve its final orientation, the first rotation is by an angle ϕ about the body z-axis, the second rotation by an angle $\theta \in [0, \pi]$ about the body y-axis, and the third rotation by an angle ψ about the body x-axis. The three rotations are illustrated in the next figure, where x'_B-y'_B-z'_B and x''_B-y''_B-z''_B are the two intermediate configurations of the body frame x_B-y_B-z_B.



Now we derive the homogeneous transformation matrix. Consider a point p in the body frame $x_{\mathcal{B}}$-$y_{\mathcal{B}}$-$z_{\mathcal{B}}$. Let us determine its coordinates in the original frame x-y-z by considering the four frames backward: $x_{\mathcal{B}}$-$y_{\mathcal{B}}$-$z_{\mathcal{B}}$, x'_B-y'_B-z'_B, x''_B-y''_B-z''_B, and x-y-z. Since the frame $x_{\mathcal{B}}$-$y_{\mathcal{B}}$-$z_{\mathcal{B}}$ is obtained from the frame x'_B-y'_B-z'_B by a rotation about the $x'_{\mathcal{B}}$-axis through an angle ψ, the coordinates of p in the latter frame is $\mathrm{Rot}_x(\psi) \cdot p$. Next, the frame x'_B-y'_B-z'_B is obtained from the frame x''_B-y''_B-z''_B after a rotation about the $y''_{\mathcal{B}}$-axis through an angle θ. So the same point has coordinates $\mathrm{Rot}_y(\theta)(\mathrm{Rot}_x(\psi) \cdot p)$ in the frame x''_B-y''_B-z''_B. Similarly, the coordinates of the point in the frame x-y-z is $\mathrm{Rot}_z(\phi)(\mathrm{Rot}_y(\theta)(\mathrm{Rot}_x(\psi) \cdot p))$. Thus the transformation matrix associated with the Z-Y-Z Euler angles are

$$
\begin{aligned}
\mathrm{Rot}_{zyx}(\phi, \theta, \psi) &= \mathrm{Rot}_z(\phi) \cdot \mathrm{Rot}_y(\theta) \cdot \mathrm{Rot}_x(\psi) \\
&= \begin{pmatrix} \cos\phi & -\sin\phi & 0 & 0 \\ \sin\phi & \cos\phi & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \cos\theta & 0 & \sin\theta & 0 \\ 0 & 1 & 0 & 0 \\ -\sin\theta & 0 & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\psi & -\sin\psi & 0 \\ 0 & \sin\psi & \cos\psi & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}
\end{aligned}
$$

$$= \begin{pmatrix} \cos\phi\cos\theta & \cos\phi\sin\theta\sin\psi - \sin\phi\cos\psi & \cos\phi\sin\theta\cos\psi + \sin\phi\sin\psi & 0 \\ \sin\phi\cos\theta & \sin\phi\sin\theta\sin\psi + \cos\phi\cos\psi & \sin\phi\sin\theta\cos\psi - \cos\phi\sin\psi & 0 \\ -\sin\theta & \cos\theta\sin\psi & \cos\theta\cos\psi & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Given the rotation matrix, we can extract the Euler angles ϕ, θ, and ψ.

References

[1] D. Marsh. *Applied Geometry for Computer Graphics and CAD*. Springer-Verlag, 1999.

[2] J. Craig. *Introduction to Robotics: Mechanics and ControL*. 2nd ed., Addison-Wesley, 1989.