Homogeneous Coordinates

Com S 477/577

Aug 26, 2008

1 Introduction

In many applications it is desirable to apply more than one transformations to an object. In robotics, for instance, an object has often undergone both translation and rotation after being manipulated. In vision, the image of an object results from a projection[1] of its model after some translation and rotation. It would be nice to concatenate all transformations into one equivalent transformation for the convenience of computation.

Example 1. A point $\boldsymbol{p} = (x, y)$ undergoes a rotation about the origin through an angle $\frac{\pi}{3}$. The resulting point $\boldsymbol{p}'$ has coordinates

$$\begin{aligned} \begin{pmatrix} x' \\ y' \end{pmatrix} &= \begin{pmatrix} \cos\frac{\pi}{3} & -\sin\frac{\pi}{3} \\ \sin\frac{\pi}{3} & \cos\frac{\pi}{3} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \\ &= \begin{pmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}. \end{aligned}$$

Next, apply to $\boldsymbol{p}'$ a shear about the origin of factor 2 in the direction of the unit vector $(\frac{\sqrt{3}}{2}, -\frac{1}{2})$. The new point $\boldsymbol{p}''$ has coordinates

$$\begin{aligned} \begin{pmatrix} x'' \\ y'' \end{pmatrix} &= \begin{pmatrix} 1 - 2\frac{\sqrt{3}}{2}\left(-\frac{1}{2}\right) & 2\left(\frac{\sqrt{3}}{2}\right)^2 \\ -2\cdot\left(-\frac{1}{2}\right)^2 & 1 + 2\frac{\sqrt{3}}{2}\left(-\frac{1}{2}\right) \end{pmatrix} \begin{pmatrix} x' \\ y' \end{pmatrix} \\ &= \begin{pmatrix} 1 + \frac{\sqrt{3}}{2} & \frac{3}{2} \\ -\frac{1}{2} & 1 - \frac{\sqrt{3}}{2} \end{pmatrix} \begin{pmatrix} x' \\ y' \end{pmatrix} \\ &= \begin{pmatrix} 1 + \frac{\sqrt{3}}{2} & \frac{3}{2} \\ -\frac{1}{2} & 1 - \frac{\sqrt{3}}{2} \end{pmatrix} \begin{pmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \\ &= \begin{pmatrix} \frac{1}{2} + \sqrt{3} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} - 1 & \frac{1}{2} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}. \end{aligned}$$

The concatenated transformation from $\boldsymbol{p}$ to $\boldsymbol{p}''$ is thus represented by the matrix

$$\begin{pmatrix} \frac{1}{2} + \sqrt{3} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} - 1 & \frac{1}{2} \end{pmatrix}.$$

[1] A projection is more general than affine transformation.

A problem is encountered when translations are involved. We would need to combine a matrix addition for the translation with a matrix multiplication for the other transformations. This is very awkward. There is a remedy, though, with the introduction of homogeneous coordinates. Then all transformations will be represented by matrices, and performed by matrix multiplications. And concatenation of transformations will be represented by the matrix product of the transformation matrices. Furthermore, an *inverse transformation*, which maps every image point back to its original position, will be obtained by taking a matrix inverse.

Recall that an affine transformation maps the point $\boldsymbol{p} = \binom{x}{y}$ to $\boldsymbol{p}' = \binom{x'}{y'} = A\boldsymbol{p} + \boldsymbol{b}$. To represent the mapping as a matrix multiplication, we introduce the homogeneous coordinates $(x, y, 1)$ such that

$$\begin{aligned} \begin{pmatrix} x' \\ y' \\ 1 \end{pmatrix} &= \begin{pmatrix} A\boldsymbol{p} + \boldsymbol{b} \\ 1 \end{pmatrix} \\ &= \begin{pmatrix} A & \boldsymbol{b} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ 1 \end{pmatrix} \end{aligned}$$

We can easily verify that the approach of homogeneous coordinates also works for the other linear transformations we have learned so far: scaling, rotation, reflection, and shear. All we need is to let $\boldsymbol{b} = 0$ in the new 3×3 transformation matrix. The problem is solved! But there is more about homogeneous coordinates we ought to know.

2 Definitions

To formally introduce homogeneous coordinates, let us first recall that a *relation* $\sim$ on a set S is a subset of $S \times S$ such that u is related to v whenever (u, v) is in the subset. We also write $u \sim v$ when u and v are related. A relation $\sim$ is *reflexive* if $u \sim u$ for all $u \in S$; it is *symmetric* if $u \sim v$ whenever $v \sim u$; it is *transitive* if $u \sim w$ whenever $u \sim v$ and $v \sim w$. The relation $\sim$ is an *equivalence* if it is reflexive, symmetric, and transitive.

The relation $>$ on $\mathbb{R}$ is transitive, but not reflexive or symmetric. The relations $\leq$ and $\geq$ are both reflexive and transitive, but not symmetric. The most familiar equivalence relation on $\mathbb{R}$ is $=$. An equivalence relation on the set $\mathbb{Z}$ of integers is congruence $\equiv$ modulo integer $m > 0$.

Let $\sim$ be an equivalence relation on S. The subset of S consisting of all elements related to an element s is the *equivalent class* of s and denoted as $[s]$. For example, the congruence $\equiv$ (mod4) induces four equivalence classes $[0], [1], [2]$, and $[3]$, where $[i] = \{\, i + 4k \mid k \text{ integer} \,\}$ for $i = 0, 1, 2, 3$.

Let us now focus on the relation $\sim$ on the set $S = \mathbb{R}^3 \setminus \{(0, 0, 0)\}$ defined below:

$$(x, y, z) \sim (u, v, w) \qquad \text{iff} \qquad (x, y, z) = r(u, v, w) \qquad \text{for some } r \neq 0. \tag{1}$$

It is easy to show that the relation is reflexive, symmetric, and transitive. Hence it is an equivalence relation. The equivalence class of (x, y, z) is the set

$$[(x, y, z)] = \{\, r(x, y, z) \mid r \in \mathbb{R} \text{ and } r \neq 0 \,\}.$$

Homogeneous coordinates are equivalence classes of the relation $\sim$ defined by (1). The coordinates (x, y, z) is identified with $r(x, y, z)$ with $r \neq 0$. The *projective plane* $\mathbb{P}^2$ is defined to be the set of all equivalence classes, that is, $\{ [(x, y, z)] \mid x \neq 0, y \neq 0, \text{ or } z \neq 0 \}$. An equivalence class is referred to as a *point* in the projective plane.

Operations of the projective plane are carried out by taking a representative from each equivalence class. For a class $[(u, v, w)]$ with $w \neq 0$, we use $(u/w, v/w, 1)$ as the representative. So there is a one-to-one correspondence between points (x, y) of the Cartesian plane and points $[(u, v, w)]$, $w \neq 0$ in the projective plane with $w \neq 0$.

3 Points at Infinity

Homogeneous coordinates of the form $(x, y, 0)$ do not correspond to a point in the Cartesian plane. Instead, they correspond to the unique point at infinity in the direction (x, y). To see this, consider the line through a point, say, (a, b), and with direction (x, y). It has the parametric form $(a+tx, b+ty)$. Every point on the line thus has homogeneous coordinates $(a + tx, b + ty, 1)$, and equivalently, $(\frac{a}{t} + x, \frac{b}{t} + y, \frac{1}{t})$. As t tends to infinity, that is, as a point moves along the line to infinity, the latter homogeneous coordinates become $(x, y, 0)$. Moving in the directions of (x, y) and $(-x, -y)$ will end up at the same infinity point represented by $(x, y, 0)$ as well as $(-x, -y, 0)$.

Hence the projective plane $\mathbb{P}^2$ can be seen as the plane $\mathbb{R}^2$ plus all the points at infinity, each of which along a different direction. The plane $\mathbb{P}^2$ also makes sense of the notion that two parallel lines intersect at infinity, as we will see in the example below.

Example 2. Consider the parallel lines $x + 2y = 1$ and $x + 2y = 2$. Let (u, v, w) be the homogeneous coordinates of a point (x, y) on the first line. Then $x = \frac{u}{w}$ and $y = \frac{v}{w}$. We have $\frac{u}{w} + \frac{2v}{w} = 1$ and thus the homogeneous equation of the line is

$$u + 2v = w. \tag{2}$$

Similarly, the second line has the homogeneous equation

$$u + 2v = 2w. \tag{3}$$

Equations (2) and (3) have solutions of the form $(-2r, r, 0)$. That is, the solutions are all homogeneous coordinates of the point $(-2, 1, 0)$, which is the unique intersection of the two parallel lines in the direction $(-2, 1)$. Similarly, we conclude that all lines parallel to $x + 2y = 1$ intersect in a unique point at infinity in the direction $(-2, 1)$.

4 Visualization of the Projective Plane

Two models exist for us to visualize the projective plane and understand homogeneous coordinates geometrically. They are the line model and the spherical model.

The line model represents the point with homogeneous coordinates $\lambda(u, v, w)$, $\lambda \neq 0$ by the line through the origin with direction (u, v, w). A one-to-one correspondence exists between the points (x, y) in the plane $\mathbb{R}^2$ and the lines parametrized as $t(x, y, 1)$, $t \in \mathbb{R}$. Another one-to-one correspondence exists between the point (x, y) and the point $(x, y, 1)$ in the plane $w = 1$.

But points at infinity, which have homogeneous coordinates of the form $(x, y, 0)$, are not on the $w = 1$ plane. Instead, they correspond to lines in the $w = 0$ plane.

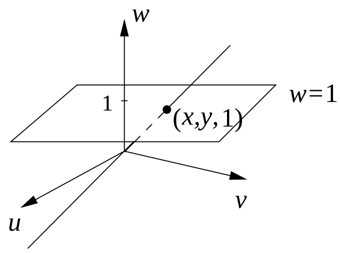

Figure 1: The line model of the projective plane.

Lines in the Cartesian plane $\mathbb{R}^2$ correspond to planes in the u-v-w space. This is the difficulty with the line model.

Example 3. The two parallel lines $x + 2y = 1$ and $x + 2y = 2$ in the previous example correspond to the planes in the u-v-w space given by equations

$$\begin{aligned} u + 2v &= w, \\ u + 2v &= 2w. \end{aligned}$$

The planes intersect in a line $(-2t, t, 0)$ through the origin in the $w = 0$ plane. This line corresponds to the point at infinity $(-2, 1, 0)$, which is intersection of the two parallel lines.

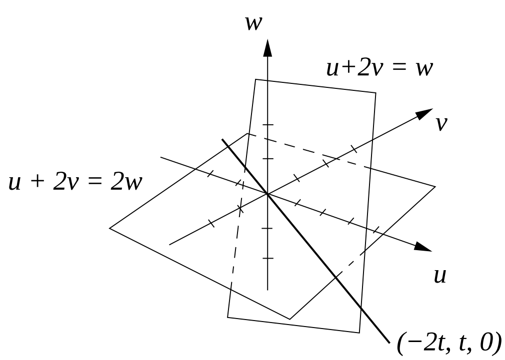

Figure 2: Intersection of plane images of parallel lines in the Cartesian plane.

In the spherical model of the projective plane, a point with homogeneous coordinates (x, y, z) maps to the point of the intersection of the corresponding line $t(x, y, z)$ and the unit sphere centered at the origin $x^2 + y^2 + z^2 = 1$. In other words,

$$(x, y, z) \mapsto \pm \frac{(x, y, z)}{\sqrt{x^2 + y^2 + z^2}}. \tag{4}$$

The two image points are antipodal. Since the two antipodal points correspond to the same point $[(x, y, z)]$ in the projective plane, it suffices to consider the upper half sphere together with (half of) the equator.

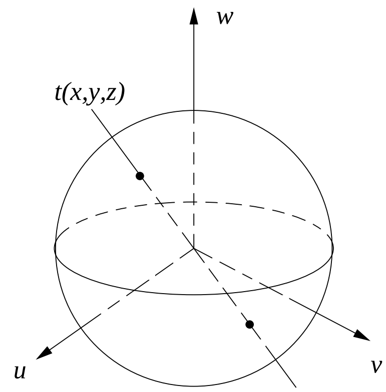

Figure 3: Spherical model of the projective plane. Antipodal points represent the same homogeneous point.

Every point at infinity in the Cartesian plane $\mathbb{R}^2$ has homogeneous coordinates $(x, y, 0)$. By (4) it corresponds to two antipodal points on the equator. Thus the sphere (or half sphere) provides a way of visualizing all homogeneous coordinates. A line in the plane $\mathbb{R}^2$ corresponds to a great circle (which is the intersection of the sphere with the plane containing the origin and the line elevated to the $z = 1$ plane). The intersection of two parallel lines correspond to the intersection points of the two great circles on the sphere, namely, two antipodal points on the equator (which represent two points at infinity in the direction of these lines).

5 Point and Line Geometry in Homogeneous Coordinates

We have seen that a point (x, y) in the Cartesian plane has homogeneous coordinates $t(x, y, 1)$, $t \neq 0$. These coordinates would correspond to a line through the origin (excluded) if they were Cartesian coordinates in the 3-dimensional space. When homogeneous coordinates are "viewed" as Cartesian coordinates, the dimensions of the geometric object they describe "increase" by 1.

A line in the Cartesian plane has general equation $ax + by + c = 0$. Suppose (u, v, w) are the homogeneous coordinates of a point (x, y) on the line; hence $x = u/w$ and $y = v/w$. Substituting for x and y in the line equation and multiplying through by w, yields the conditions for (u, v, w) to be the homogeneous coordinates of a point on the line:

$$au + bv + cw = 0. \tag{5}$$

Equation (5) is known as the *homogeneous line equation*.[2] The line is uniquely specified by the coefficients a, b, and c, or any multiple ra, rb, and rc with $r \neq 0$. Therefore it is natural to specify the line by the homogeneous coordinates

$$\ell = (a, b, c).$$

Since any non-zero multiple of ℓ defines the same line, it is useful to consider ℓ as a vector of which only the direction matters. Let $\boldsymbol{p} = (u, v, w)$ be a point in homogeneous coordinates. Then in order for $\boldsymbol{p}$ to lie on the line, the dot product of $\boldsymbol{p}$ and ℓ must vanish, that is,

$$\boldsymbol{p} \cdot \ell = 0. \tag{6}$$

[2]In Cartesian coordinates the same equation would describe a plane through the origin and with normal (a, b, c) .

The identity (6) allows us to easily determine the line through two distinct points as well as the point of intersection of two lines.

Suppose ℓ is the vector that represents a line through two distinct points $\boldsymbol{p}_1$ and $\boldsymbol{p}_2$, all in homogeneous coordinates. Then from (6) we have

$$\boldsymbol{p}_1 \cdot \ell = 0 \qquad \text{and} \qquad \boldsymbol{p}_2 \cdot \ell = 0.$$

Thus ℓ is perpendicular (or orthogonal) to both $\boldsymbol{p}_1$ and $\boldsymbol{p}_2$. To determine ℓ it suffices to find a vector perpendicular to $\boldsymbol{p}_1$ and $\boldsymbol{p}_2$ since only the direction matters. We choose the cross product by letting $\ell = \boldsymbol{p}_1 \times \boldsymbol{p}_2$ (or any multiple of $\boldsymbol{p}_1 \times \boldsymbol{p}_2$). *The equation of the line through two points can be determined by taking the 'cross product' of their homogeneous coordinates.*

Example 4. The line ℓ passing through $(3,1)$ and $(-4,5)$ satisfies equations

$$\begin{aligned} \ell \cdot (3,1,1) &= 0, \\ \ell \cdot (-4,5,1) &= 0. \end{aligned}$$

Hence we have the line in homogeneous coordinates:

$$\ell = (3,1,1) \times (-4,5,1) = (-4,-7,19),$$

which give the line $4x + 7y - 19 = 0$. We can verify this equation using the original points $(3,1)$ and $(-4,5)$.

Next, suppose $\boldsymbol{p}$ is the intersection of two lines ℓ_1 and ℓ_2 (all in homogeneous coordinates). Then from (6) we have

$$\ell_1 \cdot \boldsymbol{p} = 0 \qquad \text{and} \qquad \ell_2 \cdot \boldsymbol{p} = 0.$$

In other words, $\boldsymbol{p}$ is orthogonal to both ℓ_1 and ℓ_2 when all are seen as vectors. Hence it is sufficient to take $\boldsymbol{p} = \ell_1 \times \ell_2$ (or any multiple of it) as the homogeneous coordinates of the point of intersection.

Example 5. The intersection point $\boldsymbol{p}$ of the lines $x - 7y + 8 = 0$ and $3x - 4y + 1 = 0$ satisfies

$$(1,-7,8) \cdot \boldsymbol{p} = 0 \qquad \text{and} \qquad (3,-4,1) \cdot \boldsymbol{p} = 0.$$

Hence

$$\boldsymbol{p} = (1,-7,8) \times (3,-4,1) = (25,23,17).$$

And the two lines intersects at the point $(\frac{25}{17}, \frac{23}{17})$ in the Cartesian plane.

Example 6. The two parallel lines $2x - 5y = 0$ and $2x - 5y = -3$ do not intersect in the Cartesian plane. In homogeneous coordinates, their intersection point $\boldsymbol{p}$ is $(2,-5,0) \times (2,-5,3) = (-15,-6,0)$, which is at infinity.

References

[1] D. Marsh. *Applied Geometry for Computer Graphics and CAD*. Springer-Verlag, 1999.