Algebraic Curves*

Com S 477/577

Sep 30, 2008

An *algebraic curve* is a curve which is determined by some polynomial equation

$$f(x, y) = \sum a_{ij} x^i y^j = 0$$

in x and y. The *degree* of the curve is the degree of the polynomial $f(x, y)$. By definition, a parametric curve $\boldsymbol{\alpha}(t) = (p(t), q(t))$, where $p(t)$ and $q(t)$ are polynomials in t, is an algebraic curve. This is because the resultant of the two polynomial equations $x = p(t)$ and $y = q(t)$ in terms of t (with x and y as coefficients) is a polynomial in x and y and of degree no more than $\max(\deg(p), \deg(q))$.

Given a curve $f(x, y) = 0$, we can often get a simplified equation by rotation and translation. The shape of the curve will not be affected. Under a rotation by angle θ about the origin, a point (x, y) becomes

$$(x, y) \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} = (x\cos\theta - y\sin\theta, x\sin\theta + y\cos\theta)$$

where the matrix on the left is referred to as the *rotation matrix*. Under a translation by (x_0, y_0), the same point (x, y) becomes $(x + x_0, y + y_0)$. By properly selecting θ, x_0, y_0 and applying the corresponding rotation and translation, every quadratic curve in the general form

$$\alpha x^2 + \beta xy + \gamma y^2 + \delta x + \epsilon y + \zeta = 0,$$

where at least one of α, β, and γ is not zero, can be transformed into one of the following *canonical* forms:

$$\begin{aligned} \frac{x^2}{a^2} + \frac{y^2}{b^2} &= 1, && \text{if } \beta^2 - 4\alpha\gamma < 0; \\ \frac{x^2}{a^2} - \frac{y^2}{b^2} &= 1, && \text{if } \beta^2 - 4\alpha\gamma > 0; \\ y^2 &= 2ax, && \text{if } \beta^2 - 4\alpha\gamma = 0. \end{aligned}$$

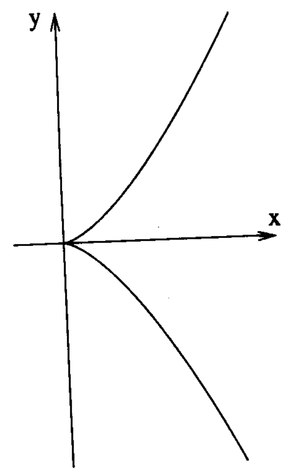
Thus all quadratic curves are classified into ellipses, hyperbolas, and parabolas[1]. For equations of higher degree there is no such simple classification. Let us first look at some examples of curves described by polynomials of degree three.

*Materials are taken from [1].

[1]These curves together are called the *conics*.

1 Some Algebraic Curves

The *semi-cubical parabola* (i.e., a cuspidal cubic) is a cubic curve which has two branches at the origin, but the tangents there to the two branches coincide. The origin is a cusp of the curve, and the common limiting tangent (i.e., the x-axis) to the two branches at the cusp is called the *cuspidal tangent*. See the figure below.



The algebraic equation of the semi-cubical parabola is

$$y^2 = x^3.$$

The parametric equation is

$$(x, y) = (t^2, t^3)$$

where t takes on any real value. This is a parameterization in that both $x(t)$ and $y(t)$ are polynomials in t. To derive this from the algebraic equation, we consider the intersection of the curve with the line $y = tx$ for each fixed value of t. Substituting $y = tx$ into $y^2 - x^3 = 0$ gives $x^2(t^2 - x) = 0$. solving this gives two copies of the point $(x, y) = 0$, together with $x = t^2$, $y = tx = t^3$. Therefore all points satisfying the algebraic equation are given by the parametric equation $(x, y) = (t^2, t^3)$.

The polar equation is

$$\rho = \sin^2\theta \sec^3\theta, \qquad -\frac{\pi}{2} < \theta < \frac{\pi}{2}.$$

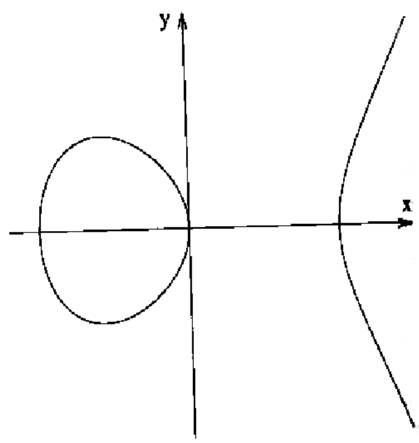
This equation is obtained by substituting $x = \rho\cos\theta$ and $y = \rho\sin\theta$ into the algebraic equation. We obtain $\rho^2(\sin^2\theta - \rho\cos^3\theta) = 0$, which gives $\rho^2 = 0$ and $\sin^2\theta - \rho\cos^3\theta = 0$. But the second equation subsumes the first one. From the polar equation we have

$$\boldsymbol{r} = (\rho\cos\theta, \rho\sin\theta) = (\tan^2\theta, \tan^3\theta),$$

and the substitution $t = \tan\theta$ yields the parametric equation.

The semi-cubical parabola is an isochronous curve; a particle descending the curve under gravity falls equal vertical distances in an equal time (Huygens 1687). It was the first algebraic curve whose arc length was calculated (Neile 1659). Before this the arc length had been found only for certain transcendental curves such as the cycloid and the equi-angular spiral.

Some plane curves appear to have several parts. The simplest example of this is the hyperbola, which has two unbounded parts that do not meet in the plane. We now give an example of a cubic curve[2] which appears to have two parts, one of which is bounded and the other of which is unbounded.



two-part cubic

The algebraic equation is

$$y^2 - x^3 + x = 0.$$

A parametric equation of the curve is

$$x = t, \quad y = \pm\sqrt{t(t^2-1)} \qquad\qquad -1 \leq t \leq 0 \text{ or } t \geq 1.$$

This parameterization by x is given by solving the algebraic equation to give y as a function of x. We determine where the line $x = t$ meets the curve for each fixed value of t. Substituting $x = t$ into $y^2 - x^3 + x = 0$ gives $y^2 - t^3 + t = 0$. Solving this gives $x = t$, $y = \pm\sqrt{t(t^2-1)}$. This gives parameterizations of four parts of the curves, one in each of the four quadrants. These parametric equations are not differentiable at $t = 0, \pm 1$.

An algebraic curve can be parametrized in many different ways. An alternative parametric equation is given by determining where the line $y = tx$ meets the curve for each fixed value of t. Substituting $y = tx$ into $y^2 - x^3 + x = 0$ gives $x(t^2x - x^2 + 1) = 0$. Solving this gives the origin $(x, y) = \mathbf{0}$, the parameterization

$$\boldsymbol{r}(t) = \frac{1}{2}\left(t^2 - \sqrt{t^4+4}, t\left(t^2 - \sqrt{t^4+4}\right)\right)$$

of the bounded part of the curve and the parameterization

$$\boldsymbol{r}(t) = \frac{1}{2}\left(t^2 + \sqrt{t^4+4}, t\left(t^2 + \sqrt{t^4+4}\right)\right)$$

of the unbounded part. The origin is not covered by the parameterization of the bounded part, because it corresponds to the value ∞ of the parameter, that is, to the vertical line $x = 0$. These two parameterization are differentiable for all values of t.

[2]The figure is taken from [1, p. 60].

The polar equation of the same curve is

$$-\rho^2 \cos^3\theta + \rho\sin^2\theta + \cos\theta = 0.$$

This is obtained by substituting $x = \rho\cos\theta$ and $y = \rho\sin\theta$ into the algebraic equation. There are two values of ρ corresponding to each value of θ.

2 Singular Points

A *singular point* of the algebraic curve $f(x,y) = 0$ is a point (a,b) on the curve at which

$$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial y} = 0,$$

that is at which the gradient $\nabla f = \mathbf{0}$. At a *non-singular point*, $\nabla f \neq 0$. To determine singular points, it is not sufficient to solve $\nabla f = \mathbf{0}$. We must find simultaneous solutions of $\nabla f = \mathbf{0}$ and $f = 0$; that is, we must find those 'singularities of the polynomial' which also lie on the curve.

EXAMPLE 1. Find the singular points of the circle $f(x,y) = x^2 + y^2 - 1$. From $\nabla f = (2x, 2y) = 0$ we obtain that $x = y = 0$. But $(0,0)$ does not lie on the circle, so the circle has no singular points.

For the cuspidal cubic $g(x,y) = y^2 - x^3 = 0$, we obtain that $\nabla g = (-3x^2, 2y) = 0$ if and only if $x = y = 0$. The only singular point is $(0,0)$.

3 Parameterization of Algebraic Curves

Consider an algebraic curve given by $f(x,y) = 0$, where f is a polynomial of degree $d \geq 1$. One question is 'Is it possible to parameterize the curve?' We now show that algebraic curves can be parametrized locally near non-singular points. In general, algebraic curves, or parts of them, can be parametrized either by x or by y, or by both.

A *local parameterization* of an algebraic curve near a point (a,b) on the curve, is a parameterization $J \to \mathbb{R}^2$ of a piece of the curve including the point (a,b). Usually we assume that the point (a,b) corresponds to an interior point of the interval J of parameterization.

Theorem 1 (Local Parameterization Theorem) *Let f be a polynomial of degree $d \geq 1$ satisfying $f(a,b) = 0$ and $\frac{\partial f}{\partial y}(a,b) \neq 0$. Then there is an interval $J = (a-\epsilon, a+\epsilon)$ where $\epsilon > 0$, and a unique smooth function $\phi : J \to \mathbb{R}$ such that $\phi(a) = b$ and $f(x, \phi(x)) = 0$ for all $x \in J$. This solution gives a unique regular smooth local parameterization of the curve in the neighborhood of the point (a,b) by the x coordinate, that is, $\boldsymbol{r}(x) = (x, \phi(x))$ for all $x \in (a-\epsilon, a+\epsilon)$. Locally, the curve $f(x,y) = 0$ is the graph of the function $y = \phi(x)$, and*

$$\frac{d\phi}{dx} = -\frac{f_x}{f_y}, \qquad x \in J.$$

Similarly, in the case where $\frac{\partial f}{\partial x}(a,b) \neq 0$, there exists a unique regular smooth local parameterization $(\psi(y), y)$ defined near b and satisfying $\psi(b) = a$, $f(\psi(y), y) = 0$, and

$$\frac{d\psi}{dy} = -\frac{f_y}{f_x}$$

for y near b. In the neighborhood of (a,b) the curve $f(x,y) = 0$ is the graph of the function $x = \psi(y)$.

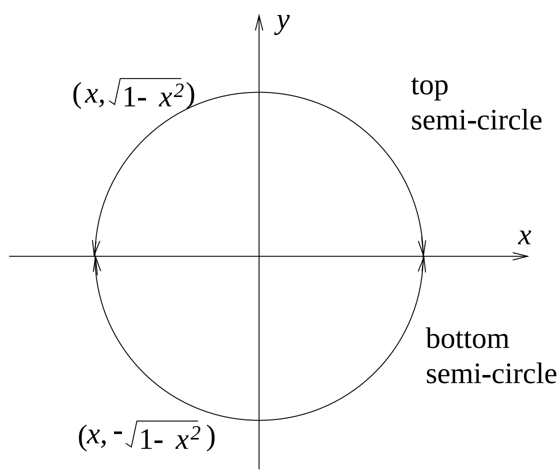
The above theorem is a special case of the implicit function theorem, of which the proof can be found in texts on analysis.

These smooth local parameterizations of algebraic curves by x or y are regular; for example, the derivative of $(x, \phi(x))$ is $(1, \phi'(x)) \neq 0$.

Example 2. Consider the circle given by $f(x,y) = x^2 + y^2 - 1 = 0$. We have $f_y = 2y \neq 0$ when $y \neq 0$. Thus the theorem tells us there is a local smooth parameterization by x near (a,b) where $a^2 + b^2 = 1$ and $b \neq 0$. Indeed the parameterization is given by

$$\left(x, \sqrt{1-x^2}\right) \qquad \text{or by} \qquad \left(x, -\sqrt{1-x^2}\right), \qquad\qquad -1 < x < 1$$

depending on whether $b > 0$ or $b < 0$. In this case, for fixed a with $-1 < a < 1$, there are two values for b, and local parameterizations are given.



Similarly, for $a \neq 0$, there is a local parameterization by y near (a,b). Again this is given by

$$\left(\sqrt{1-y^2}, y\right) \qquad \text{or by} \qquad \left(-\sqrt{1-y^2}, y\right), \qquad\qquad -1 < y < 1$$

depending on whether $a > 0$ or $a < 0$.

4 Curvature of Algebraic Curves

In general algebraic curves cannot be parametrized in a simple way, that is, by using a simple formula which gives a parameterization of the whole curve. Therefore the formula for the curvature of parametric curves cannot be used. However, according to Theorem 1 algebraic curves can be parametrized locally near a non-singular point, though not in general using a simple formula. We now use this result to give a parameter-free formula for the curvature of algebraic curves.

We recall that the Hessian matrix of a polynomial $f(x,y)$ is

$$H = \begin{pmatrix} f_{xx} & f_{xy} \\ f_{xy} & f_{yy} \end{pmatrix}.$$

Theorem 2 *The curvature of the algebraic curve $f(x,y) = 0$ at a non-singular point (x,y) on the* *curve is*

$$\begin{aligned}\kappa &= \pm\frac{(f_y,-f_x)H\begin{pmatrix} f_y \\ -f_x \end{pmatrix}}{\|\nabla f\|^3} \\ &= \pm\frac{f_y^2 f_{xx} - 2f_x f_y f_{xy} + f_x^2 f_{yy}}{(f_x^2+f_y^2)^{\frac{3}{2}}} \\ &= \pm\frac{f_y^2 f_{xx} - 2f_x f_y f_{xy} + f_x^2 f_{yy}}{(f_x^2+f_y^2)^{\frac{3}{2}}},\end{aligned}$$

where the sign '$-$' is chosen in case the motion along the curve is in the direction of the vector $(f_y, -f_x)$ and where the sign '$+$' is chosen in case the motion along the curve is in the direction of the vector $(-f_y, f_x)$.

Proof By Theorem 1, the algebraic curve can be given a regular local parameterization near a non-singular point by $\boldsymbol{\alpha}(t) = (x(t), y(t))$. Differentiating the equation

$$f\Big(x(t), y(t)\Big) = 0,$$

using the chain rule, we obtain

$$\boldsymbol{\alpha}' \cdot (f_x, f_y) = 0.$$

Therefore $\boldsymbol{\alpha}'$ is orthogonal to (f_x, f_y), and we have $\boldsymbol{\alpha}' = (x', y') = \lambda(f_y, -f_x)$ where $\lambda \neq 0$ is a function of t. On differentiating once more we obtain

$$\boldsymbol{\alpha}'' \cdot (f_x, f_y) + (x', y')H\begin{pmatrix} x' \\ y' \end{pmatrix} = 0.$$

Therefore we have

$$\begin{aligned} x'y'' - y'x'' &= \boldsymbol{\alpha}'' \cdot (-y', x') \\ &= \lambda\boldsymbol{\alpha}'' \cdot (f_x, f_y) \\ &= -\lambda(x', y')H\begin{pmatrix} x' \\ y' \end{pmatrix} \\ &= -\lambda^3(f_y, -f_x)H\begin{pmatrix} f_y \\ -f_x \end{pmatrix}. \end{aligned}$$

Since $\|\boldsymbol{\alpha}'\| = |\lambda| \cdot \|(f_y, -f_x)\|$, we have that

$$\kappa = \frac{\boldsymbol{\alpha}' \times \boldsymbol{\alpha}''}{\|\boldsymbol{\alpha}'\|^3} = \frac{x'y'' - y'x''}{\|\boldsymbol{\alpha}'\|^3} = \pm\frac{f_y^2 f_{xx} - 2f_x f_y f_{xy} + f_x^2 f_{yy}}{\|\nabla f\|^3}.$$

□

Corollary 3 *The algebraic curve has a point of inflection at a non-singular point (a, b) if and only if,*

$$f_y^2 f_{xx} - 2f_x f_y f_{xy} + f_x^2 f_{yy}$$

is zero at (a, b) and changes sign as (x, y) moves through (a, b) along the curve.

Thus in order to find all points of inflection of an algebraic curve we first determine the points where the curvature is zero by finding the simultaneous solutions of the equations

$$\begin{aligned} f(x,y) &= 0, \\ f_y^2 f_{xx} - 2f_x f_y f_{xy} + f_x^2 f_{yy} &= 0, \end{aligned}$$

and then find those solutions which are non-singular points. Next, we determine whether the curvature changes sign as we move along the curve past the point of zero curvature. If it does, the point is an inflection. In practice we can check on the change of sign if, for example, the curve can be parametrized near the point by x, and

$$f_y^2 f_{xx} - 2f_x f_y f_{xy} + f_x^2 f_{yy}$$

becomes a function of x after substituting from $f(x,y) = 0$. Such a local parameterization by x exists by Theorem 1 provided that the tangent at the point is not parallel to the y-axis.

EXAMPLE 3. Calculate the curvature of the hyperbola

$$x^2 - 3y^2 = 1$$

at the point $(2,1)$.

Let $f = x^2 - 3y^2 - 1$. We have

$$\begin{aligned} f_x &= 2x, \\ f_y &= -6y, \\ f_{xx} &= 2, \\ f_{xy} &= 0, \\ f_{yy} &= -6. \end{aligned}$$

Therefore

$$\begin{aligned} \kappa &= \pm\frac{(-6y)^2 \cdot 2 + (2x)^2 \cdot (-6)}{(4x^2 + 36y^2)^{\frac{3}{2}}} \\ &= \pm\frac{9y^2 - 3x^2}{(x^2 + 9y^2)^{\frac{3}{2}}}. \end{aligned}$$

In particular,

$$\kappa(2,1) = \pm\frac{3}{(\sqrt{13})^3}.$$

Example 4. Find the points of inflection of the acnodal cubic

$$f(x,y) = y^2 + x^2 - x^3 = 0.$$

We have

$$\begin{aligned} f_x &= 2x - 3x^2, \\ f_y &= 2y, \\ f_{xx} &= 2 - 6x, \\ f_{xy} &= 0, \\ f_{yy} &= 2. \end{aligned}$$

Eliminating y from

$$\begin{aligned}
f(x,y) &= y^2 + x^2 - x^3 \\
&= 0, \\
f_y^2 f_{xx} - 2 f_x f_y f_{xy} + f_x^2 f_{yy} &= 4y^2(2-6x) + 2(2x-3x^2)^2 \\
&= 8y^2 - 24xy^2 + 8x^2 - 24x^3 + 18x^4,
\end{aligned}$$

we have, on the curve, that

$$f_y^2 f_{xx} - 2 f_x f_y f_{xy} + f_x^2 f_{yy} = 8x^3 - 6x^4.$$

The origin is a singular point. Therefore there are precisely two points of zero curvature. They are $\left(\frac{4}{3}, \pm\frac{4}{3\sqrt{3}}\right)$. At these points

$$\nabla f = (2x - 3x^2, 2y)$$

is not parallel to the y-axis, and therefore the curve can be parametrized locally by x. Also

$$8x^3 - 6x^4 = 6x^3\left(\frac{4}{3} - x\right)$$

changes sign as we move along the curve past each of the points and therefore the curvature also changes sign. Thus the two points are points of inflection.

References

[1] J. W. Rutter. *Geometry of Curves*. Chapman & Hall/CRC, 2000.