Linear Programming

Computer Science 511

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The Fundamental Theorem of Linear Programming

Theorem

The following statements hold for any linear program in standard form.

1. *If there is no optimal solution, then the problem is either infeasible or unbounded.*
2. *If a feasible solution exists, then a basic feasible solution exists.*
3. *If an optimum solution exists, then there exists a basic feasible solution that is also an optimum solution.*

Primal

$$
\begin{aligned}
\text{maximize} \quad & \sum_{j=1}^{n} c_j x_j \\
\text{subject to} \quad & \sum_{j=1}^{n} a_{ij} x_j \leq b_i && i = 1, \ldots, m \\
& x_j \geq 0 && j = 1, \ldots n
\end{aligned}
$$

Dual

$$
\begin{aligned}
\text{minimize} \quad & \sum_{i=1}^{m} b_i y_i \\
\text{subject to} \quad & \sum_{i=1}^{m} a_{ij} y_i \geq c_j && j = 1, \ldots, n \\
& y_i \geq 0 && i = 1, \ldots m
\end{aligned}
$$

Theorem (Weak Duality)

If $(x_1, \ldots, x_n)$ is a feasible solution for the primal and $(y_1, \ldots, y_m)$ is a feasible solution for the dual, then

$$\sum_{j=1}^{n} c_j x_j \leq \sum_{i=1}^{m} b_i y_i.$$

Proof.

$$\begin{aligned}
\sum_{j=1}^{n} c_j x_j &\leq \sum_{j=1}^{n} \left(\sum_{i=1}^{m} a_{ij} y_i \right) x_j && \text{by dual feasibility} \\
&= \sum_{i=1}^{m} \left(\sum_{j=1}^{n} a_{ij} x_j \right) y_i \\
&\leq \sum_{i=1}^{m} b_i y_i && \text{by primal feasibility.}
\end{aligned}$$

□

The Strong Duality Theorem

Theorem

If the primal has optimal solution $(x_1^*, \ldots, x_n^*)$*, then the dual has an optimal solution* $(y_1^*, \ldots, y_m^*)$ *such that*

$$\sum_{j=1}^{n} c_j x_j^* = \sum_{i=1}^{m} b_i y_i^*.$$

Lemma

Suppose that the solution returned by the simplex algorithm is $(x_1^*, \ldots, x_n^*)$ *and that in the final dictionary*

$$z = z^* + \sum_{j=1}^{n} \overline{c}_j x_j + \sum_{i=1}^{m} \overline{c}_{n+i} x_{n+i}.$$

Let

$$y_i^* = -\overline{c}_{n+i}, \qquad \text{for } i = 1, \ldots, m.$$

Then $(y_1^*, \ldots, y_m^*)$ *is dual feasible and*

$$\sum_{j=1}^{n} c_j x_j^* = \sum_{i=1}^{m} b_i y_i^*.$$

Strong duality follows directly

Proof of the Key Lemma: Step 1

Substitute

$$y_i^* = -\overline{c}_{n+i} \qquad \text{and} \qquad x_{n+i} = b_i - \sum_{j=1}^{n} a_{ij} x_j$$

into

$$z = z^* + \sum_{j=1}^{n} \overline{c}_j x_j + \sum_{i=1}^{m} \overline{c}_{n+i} x_{n+i}$$

to get

$$\begin{aligned}\sum_{j=1}^{n} c_j x_j &= z^* + \sum_{j=1}^{n} \overline{c}_j x_j - \sum_{i=1}^{m} y_i^* \left(b_i - \sum_{j=1}^{n} a_{ij} x_j \right) \\ &= \left(z^* - \sum_{i=1}^{m} b_i y_i^* \right) + \sum_{j=1}^{n} \left(\overline{c}_j + \sum_{i=1}^{m} a_{ij} y_i^* \right) x_j.\end{aligned}$$

Proof of the Key Lemma: Step 2

Equate coefficients in

$$\sum_{j=1}^{n} c_j x_j = \left(z^* - \sum_{i=1}^{m} b_i y_i^* \right) + \sum_{j=1}^{n} \left(\overline{c}_j + \sum_{i=1}^{m} a_{ij} y_i^* \right) x_j$$

to get

$$z^* = \sum_{i=1}^{m} b_i y_i^*$$

and

$$c_j = \overline{c}_j + \sum_{i=1}^{m} a_{ij} y_i^*, \qquad j = 1, \ldots, n.$$

Proof of the Key Lemma: Step 3

By definition,

$$z^* = \sum_{j=1}^{n} c_j x_j^* .$$

Recall that

$$z^* = \sum_{i=1}^{m} b_i y_i^* .$$

So

$$\sum_{j=1}^{n} c_j x_j^* = \sum_{i=1}^{m} b_i y_i^* ,$$

as claimed.

Proof of the Key Lemma: Last Step

Recall that

$$c_j = \overline{c}_j + \sum_{i=1}^{m} a_{ij} y_i^*, \qquad j = 1, \ldots, n.$$

Because this is the final dictionary, $\overline{c}_k \leq 0$ for every k, so

$$\sum_{i=1}^{m} a_{ij} y_i^* \geq c_j, \qquad j = 1, \ldots, n$$

and

$$y_i^* \geq 0, \qquad i = 1, \ldots, m.$$

Therefore, y^* is dual feasible. □