Computer Science 511
Fall 2008
Exam 1

Monday, October 6

This closed-book, closed-notes two-hour test consists of 6 questions. The number of points for each problem is indicated on the next page.

- Read all questions carefully before starting.
- Work on the problems that seem easiest first.
- Attempt to solve all problems.
- Show your work, but also remember that we prefer concise answers.
- Write all your answers clearly on the space provided in the exam paper. If you need additional paper, please ask us.
- If you do not understand a problem, please ask us for clarification.
- Clearly state any simplifying assumptions you make in solving a problem.
- When asked to describe algorithm, you are expected to argue its correctness and analyze its running time.

Name: ____

Score

1	2	3	4	5	6	Total
15	15	15	15	20	20	100

1 Network Flows (15 points)

Decide whether the following statements are true or false. If true, give a short explanation; if false, give a counterexample.

(a) (7 points) Let G be an arbitrary flow network with source s, sink t, and a positive integer capacity $c(e)$ for each edge e. Then, every maximum flow in G must be integral (i.e., the flow in each edge must be integer-valued).

(b) (8 points) Let G be an arbitrary flow network with source s, sink t, and a positive integer capacity $c(e)$ for each edge e. Let us call a flow *even* if the flow on each edge is an even number. Suppose all capacities in G are even numbers. Then, G has an even maximum flow.

2 Circulations (15 points)

A police department in a small city consists of three precincts, denoted p_1, p_2, and p_3. Each precinct is assigned a number of patrol cars with two-way radios and first-aid equipment. The department operates with three shifts. The tables below show the minimum and maximum number of cars needed in each shift.

	Shift 1	Shift 2	Shift 3
p_1	2	4	3
p_2	3	6	5
p_3	5	7	6

(a)

	Shift 1	Shift 2	Shift 3
p_1	3	7	5
p_2	5	7	10
p_3	8	12	10

(b)

Table 1: (a) Minimum number of cars per shift. (b) Maximum number of cars per shift.

Administrative constraints require that (1) shifts 1, 2, and 3 have, respectively, at least 10, 20, and 18 cars available and (2) precincts p_1, p_2, and p_3 are, respectively, allocated at least 10, 14, and 13 cars.

The police department wants to determine if there exists an allocation of patrol cars that meets all the requirements. Formulate this problem as a circulation problem.

Note. A complete solution must describe the network used to solve this problem, together with its edge capacities, lower bounds, and node demands, as needed. It should also include a brief justification of the formulation. *You do not need to solve the circulation problem you formulate.*

3 Linear Programming and Flows (15 points)

Formulate each of the following problems as linear programming problems:

(a) (7 points) Find a maximum s-t flow in a network where each vertex u also has a capacity $c(u)$ on the maximum flow that can enter it. As usual, we assume that each edge e has a capacity $c(e)$ and that flow conservation and capacity constraints must be satisfied.

(b) (8 points) Let G be a flow network where every node v has a demand $d(v)$ and every edge e has a capacity $c(e)$, a lower bound $\ell(e)$, and a *cost* $w(e)$. Suppose that the cost of sending $f(e)$ units of flow along edge e is $w(e) \cdot f(e)$. The problem is to find a feasible flow (i.e., a circulation) of minimum total cost.

4 Formulating Linear Programs (15 points)

A steel company must decide how to allocate next week's time on a rolling mill, which is a machine that takes unfinished slabs of steel as input and can produce either one of two semi-finished products: bands and coils. The products come off the rolling line at two different rates:

Bands	200 tons/hour
Coils	160 tons/hour.

They also produce different profits:

Bands	\$25/ton
Coils	\$30/ton.

Based on the currently booked orders, the following upper bounds are placed on the amount of each product to produce:

Bands	6000 tons
Coils	4000 tons.

Given that there are 40 hours of production time available this week, the problem is to decide how many tons of bands and how many tons of coils should be produced to yield the greatest profit.

(a) (8 points) Formulate this problem as a linear programming problem.

(b) (4 points) Draw the feasible region.

(c) (3 points) What is the optimum solution and what is its value?

5 Linear Programming (20 points)

Consider the following linear program:

$$\begin{array}{lrcl} \text{maximize} & x - y & & \\ \text{subject to} & ax + by & \leq & -1 \\ & x, y & \geq & 0, \end{array} \tag{1}$$

where a and b are real numbers.

(a) (5 points) Under what conditions on the values of a and b is linear program (1) infeasible? Justify your answer.

(b) (5 points) Under what conditions on the values of a and b is linear program (1) unbounded? Justify your answer.

(c) (5 points) Under what conditions on the values of a and b does linear program (1) have a finite and unique optimal solution? Justify your answer.

(d) (5 points) Write the dual of linear program (1).

6 Network Flows (20 points)

Let $G = (V, E)$ be a directed graph with specified source and sink vertices s and t, respectively. Two directed $s \to t$ paths is *vertex-disjoint* if they share no vertices other than s and t.

(a) (12 points) Give a polynomial-time algorithm that finds the maximum number of vertex-disjoint $s \to t$ paths in G.

A subset $S \subseteq V - \{s, t\}$ is an s-t *separator* in G if every directed path from s to t in G contains a vertex in S.

(b) (8 points) Give a polynomial-time algorithm that finds an s-t separator in G with the minimum number of vertices.