Tree Decompositions and Tree-Width

CS 511

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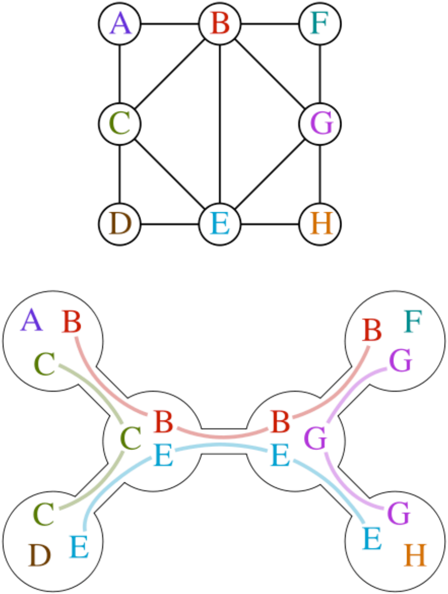
November 17, 2008

Tree Decompositions

Definition

A tree decomposition of a graph $G = (V, E)$ consists of a tree T and a subset $V_t \subseteq V$ for every node $t \in T$, such that the collection $\{V_t : t \in T\}$ satisfies:

- (Node coverage) For every $v \in V$, there is some node t in T such that $v \in V_t$.
- (Edge coverage) For every $e \in E$, there is some node t in T such that V_t contains both endpoints of e.
- (Coherence) Let t_1, t_2, t_3 be three nodes in T such that t_2 lies on the path between t_1 and t_3 in T. Then, if $v \in V$ belongs to both V_{t_1} and V_{t_3}, v must also belong to V_{t_2}.



Author: David Eppstein. Source: Wikipedia.

Tree-Width

Definition

The width of tree decomposition $(T, \{V_t : t \in T\})$ is

$$\text{width}(T, \{V_t : t \in T\}) = \max_{t \in T} |V_t| - 1.$$

Definition

The tree-width of G, denoted $\text{tw}(G)$, is the minimum width of a tree decomposition of G.

Notation

Let $(T, \{V_t : t \in T\})$ be a tree decomposition of G. Then, if T' is a subgraph of T, $G_{T'}$ denotes the subgraph induced by the set $\bigcup_{t \in T'} V_t$.

Theorem (Node Separation Property)

Suppose $T - t$ has components $T_1, \ldots, T_d$. Then, the subgraphs

$$G_{T_1} - V_t, G_{T_2} - V_t, \ldots, G_{T_d} - V_t$$

have no nodes in common, and there are no edges between them.

Theorem (Edge Separation Property)

Let X and Y be the two components of T after the deletion of edge (x, y). Then, deleting $V_x \cap V_y$ disconnects G into two subgraphs $H_X = G_X - (V_x \cap V_y)$ and $H_Y = G_Y - (V_x \cap V_y)$. That is,

- *H_X and H_Y share no nodes and*
- *there is no edge in G with one endpoint in H_X and the other in H_Y.*

Definition

A tree decomposition $(T, \{V_t : t \in T\})$ of G is nonredundant if there is no edge (x, y) in T such that $V_x \subseteq V_y$.

Lemma

Any graph has a nonredundant tree decomposition.

Lemma

Any non-redundant tree decomposition of an n-node graph has at most n pieces.

Rooted tree decomposition

Definition

A rooted tree decomposition of G is a tree decomposition $(T, \{V_t : t \in T\})$ of G where some node r in T is declared to be the root.

Let t be a node in a rooted tree decomposition. Then,

- T_t is the subtree of T rooted at t,
- G_t is the subgraph of G induced by the vertices in $\bigcup_{x \in T_t} V_x$.

Subproblems

Definition

For each node t in a rooted tree decomposition of G and each independent set $U \subseteq V_t$, $\text{opt}_U(t)$ is the maximum weight of an independent set S of G_t such that $S \cap V_t = U$.

Optimal Substructure

Let

- t be a node in T with children $t_1, \ldots, t_d$,
- U be an independent set of V_t,
- S be a maximum independent set in G_t subject to $S \cap V_t = U$ (i.e., $w(S) = \text{opt}_U(t)$),
- S_i be the intersection of S with the nodes of G_{T_i}.

Lemma (Optimal Substructure)

S_i is a maximum-weight independent set of G_{t_i}, subject to the constraint that $S_i \cap V_t = U \cap V_{t_i}$.

A Recurrence for Maximum Weight Independent Set

Theorem

The value of $\text{opt}_U(t)$ *is given by*

$$\text{opt}_U(t) = w(U) + \sum_{i=1}^{d} \max\{\text{opt}_{U_i}(t_i) - w(U_i \cap U) : U_i \subseteq V_{t_i} \text{ is independent and } U_i \cap V_t = U \cap V_{t_i}\}.$$