First-Order Logic

Outline

- Why FOL?
- Syntax and semantics of FOL
- Using FOL
- Wumpus world in FOL
- Knowledge engineering in FOL

2

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Pros and cons of propositional logic

- ☺ Propositional logic is declarative
  - Contrast with procedural approach (Programming languages C++, Java)
  - knowledge and inference are separate, inference is entirely domain-independent
- ☺ Propositional logic allows partial/disjunctive/negated information
  - (unlike most data structures and databases)
- ☺ Propositional logic is compositional:
  - meaning of $B_{1,1} \wedge P_{1,2}$ is derived from meaning of $B_{1,1}$ and of $P_{1,2}$
- ☺ Meaning in propositional logic is context-independent
  - (unlike natural language, where meaning depends on context)
- ☹ Propositional logic has very limited expressive power
  - (unlike natural language)
  - E.g., cannot say "pits cause breezes in adjacent squares“
    - except by writing one sentence for each square

3

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First-order logic

- Whereas propositional logic assumes the world contains facts,
  - PL : facts hold or do not hold.
- First-order logic (like natural language) assumes the world contains
  - Objects: people, houses, numbers, colors, baseball games, wars, …
  - Relations: red, round, prime, brother of, bigger than, part of, comes between, …
  - Functions: father of, best friend, one more than, plus, …
- FOL : objects with relations between them that hold or do not hold

4

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Syntax of FOL: Basic elements

- Constants John, 2, ISU,...
- Predicates Brother, Teacher, >,...
- Functions Sqrt, LeftLegOf,...
- Variables x, y, a, b,...
- Connectives $\neg$, $\Rightarrow$, $\wedge$, $\vee$, $\Leftrightarrow$
- Equality =
- Quantifiers $\forall$, $\exists$

5

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Atomic sentences

Atomic sentence = *predicate* ($term_1$,...,$term_n$) or $term_1$ = $term_2$

Term = *function* ($term_1$,...,$term_n$) or *constant* or *variable*

(A term refers to an object)

- E.g., *Brother(KingJohn,RichardTheLionheart)*

Length(LeftLegOf(Richard))=Length(LeftLegOf(KingJohn))

6

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Complex sentences

- Complex sentences are made from atomic sentences using connectives

  $\neg S$, $S_1 \land S_2$, $S_1 \lor S_2$, $S_1 \Rightarrow S_2$, $S_1 \Leftrightarrow S_2$,

E.g. *Sibling(KingJohn,Richard)* $\Rightarrow$ *Sibling(Richard,KingJohn)*

$>(1,2) \lor \leq (1,2)$

$>(1,2) \land \neg >(1,2)$

7

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Complex sentences

- Universal quantification

  $\forall$<*variables*> <*sentence*>

  Everyone at ISU is smart:

  $\forall$x At(x,ISU) $\Rightarrow$ Smart(x)

- Existential quantification

  $\exists$<*variables*> <*sentence*>

  Someone at ISU is smart:

  $\exists x$ At(x, ISU) $\land$ Smart(x)

8

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Truth in first-order logic

- Sentences are true with respect to a model
- Model contains objects (called domain elements) and an interpretation of symbols
- Interpretation specifies referents for
  constant symbol → object in domain D
  predicate symbol → relation
  function symbol → functional relation
- Each predicate symbol of arity k is mapped to a relation {<object1, ..., objectk>}, a set of k-tuples over D^k which are true or, equivalently, a function from D^k to {*true*,*false*}
- Each function symbol of arity k is mapped to a function from D^k to D+1 (domain + an invisible object)

9

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Truth in first-order logic

- An atomic sentence *predicate*($term_1,\ldots,term_n$) is true
  iff the objects referred to by $term_1,\ldots,term_n$
  are in the relation referred to by *predicate*.
- $term_1 = term_2$ is true under a given interpretation if and only if $term_1$ and $term_2$ refer to the same object
- The semantics of sentences formed with logical connectives is identical to that in PL

10

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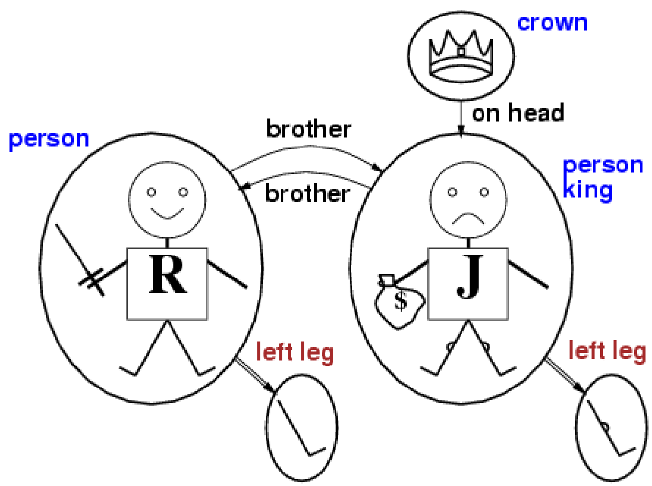
Models for FOL: Example



11

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Models for FOL: Example

- The intended interpretation
  - Constant Richard → Richard the Lionheart
  - Constant John → King John
  - Predicate Brother → the brotherhood relation
    {< Richard the Lionheart, King John>, <King John, Richard the Lionheart>}
  - Predicates OnHead, Person, King, Crown …
  - Function Leftleg ->
    <Richard the Lionheart> → Richard's left leg
    < King John> → John's left leg

- Another interpretation
  - Constant Richard → the crown
  - Constant John → King John's left leg
  - Predicate Brother →
    {< Richard the Lionheart, the crown>}
  - …

12

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Models for FOL

- Entailment can be defined
- We can enumerate the models for a given KB vocabulary:

For each number of domain elements n from 1 to ∞
  For each k-ary predicate P_k in the vocabulary
    For each possible k-ary relation on n objects
      For each constant symbol C in the vocabulary
        For each choice of referent for C from n objects . . .

- Computing entailment by enumerating the models will not be easy !!
  - the number of possible models may be unbounded or very large (10^{25} for the example)

13

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Quantifiers

- Allows us to express properties of collections of objects instead of enumerating objects by name

- Universal: “for all” $\forall$
- Existential: “there exists” $\exists$

14

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Universal quantification

∀<*variables*> <*sentence*>

Everyone at ISU is smart:

$\forall x\ At(x,ISU) \Rightarrow Smart(x)$

$\forall x\ P$ is true in a model *m* iff *P* is true with *x* being each possible object in the model

Roughly speaking, equivalent to the conjunction of instantiations of *P*

$At(KingJohn,ISU) \Rightarrow Smart(KingJohn)$
$\land\ At(Richard, ISU) \Rightarrow Smart(Richard)$
$\land\ At(ISU,ISU) \Rightarrow Smart(ISU)$
$\land\ \ldots$

A common mistake to avoid

- Typically, $\Rightarrow$ is the main connective with $\forall$

- Common mistake: using $\land$ as the main connective with $\forall$:

  $\forall x\ At(x, ISU) \land Smart(x)$

  means “Everyone is at ISU and everyone is smart”

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Existential quantification

∃<*variables*> <*sentence*>

Someone at ISU is smart:
$\exists x$ At(x, ISU) $\wedge$ Smart(x)

$\exists x\ P$ is true in a model *m* iff *P* is true with *x* being some possible object in the model

- Roughly speaking, equivalent to the disjunction of instantiations of *P*

  At(KingJohn,ISU) $\wedge$ Smart(KingJohn)
  $\vee$ At(Richard,ISU) $\wedge$ Smart(Richard)
  $\vee$ At(ISU, ISU) $\wedge$ Smart(ISU)
  $\vee$ ...

Another common mistake to avoid

- Typically, $\wedge$ is the natural connective with $\exists$
- Common mistake: using $\Rightarrow$ as the main connective with $\exists$:

  $$\exists x\ \text{At}(x, \text{ISU}) \Rightarrow \text{Smart}(x)$$

  is true if there is anyone who is not at ISU!

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Properties of quantifiers

∀x ∀y is the same as ∀y ∀x, ∀y,x
∃x ∃y is the same as ∃y ∃x, ∃y,x

∃x ∀y is not the same as ∀y ∃x
∃x ∀y Loves(x,y)
- "There is a person who loves everyone in the world"

∀y ∃x Loves(x,y)
- "Everyone in the world is loved by at least one person"

- Quantifier duality: each can be expressed using the other
  ∀x Likes(x,IceCream)   ¬∃x ¬Likes(x,IceCream)
  ∃x Likes(x,Broccoli)   ¬∀x ¬Likes(x,Broccoli)

19

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Using FOL

The kinship domain:
Predicates *Parent, Brother, Sibling, Child, Son, Wife*, *Cousin*, ...

- Brothers are siblings
  ∀x,y *Brother(x,y)* ⇒ *Sibling(x,y)*
- One's mother is one's female parent
  ∀m,c *Mother(c)* = m ⇔ *(Female(m)* ∧ *Parent(m,c))*
  (use Mother(m,c)?)
- "Sibling" is symmetric

20

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Using FOL

The kinship domain:

- Brothers are siblings
  $\forall x,y\ Brother(x,y) \Rightarrow Sibling(x,y)$
- One's mother is one's female parent
  $\forall m,c\ Mother(c) = m \Leftrightarrow (Female(m) \wedge Parent(m,c))$
- "Sibling" is symmetric
  $\forall x,y\ Sibling(x,y) \Leftrightarrow Sibling(y,x)$
- A first cousin is a child of a parent's sibling

Using FOL

The kinship domain:

- Brothers are siblings
  $\forall x,y\ Brother(x,y) \Rightarrow Sibling(x,y)$
- One's mother is one's female parent
  $\forall m,c\ Mother(c) = m \Leftrightarrow (Female(m) \wedge Parent(m,c))$
- "Sibling" is symmetric
  $\forall x,y\ Sibling(x,y) \Leftrightarrow Sibling(y,x)$
- A first cousin is a child of a parent's sibling
  $\forall x,y\ FirstCousin(x,y) \Leftrightarrow \exists p,ps\ Parent(p,x) \wedge Sibling(ps,p) \wedge Parent(ps,y)$
- Definition of full *Sibling* in terms of *Parent*:

Using FOL

The kinship domain:

- Brothers are siblings
  $\forall x,y\ Brother(x,y) \Rightarrow Sibling(x,y)$
- One's mother is one's female parent
  $\forall m,c\ Mother(c) = m \Leftrightarrow (Female(m) \wedge Parent(m,c))$
- “Sibling” is symmetric
  $\forall x,y\ Sibling(x,y) \Leftrightarrow Sibling(y,x)$
- A first cousin is a child of a parent’s sibling
  $\forall x,y\ FirstCousin(x,y) \Leftrightarrow \exists p,ps\ Parent(p,x) \wedge Sibling(ps,p) \wedge Parent(ps,y)$

- Definition of full *Sibling* in terms of *Parent*:
  $\forall x,y\ FullSibling(x,y) \Leftrightarrow [\neg(x = y) \wedge \exists m,f\ \neg(m = f) \wedge Parent(m,x) \wedge Parent(f,x) \wedge Parent(m,y) \wedge Parent(f,y)]$

Using FOL - The set domain

The set theory can be built up from a tiny kernel of axioms

Constant { }, unary predicate *Set*
Syntactic sugar: an extension to the standard syntax to make sentences easier to read

- Binary functions: $\{x|s\}$, $s_1 \cap s_2$, $s_1 \cup s_2$
- Binary predicates: $x \in s$, $s_1 \subseteq s_2$

- $\forall s\ Set(s) \Leftrightarrow (s = \{\}) \vee (\exists x,s_2\ Set(s_2) \wedge [s = \{x|s_2\}])$
- $\neg\exists x,s\quad \{x|s\} = \{\}$
- $\forall x,s\quad x \in s \Leftrightarrow s = \{x|s\}$
- $\forall x,s\quad x \in s \Leftrightarrow [\exists y,s_2\ (s = \{y|s_2\} \wedge (x = y \vee x \in s_2))]$
- $\forall s_1,s_2\quad s_1 \subseteq s_2 \Leftrightarrow (\forall x\ \ x \in s_1 \Rightarrow x \in s_2)$
- $\forall s_1,s_2\quad (s_1 = s_2) \Leftrightarrow (s_1 \subseteq s_2 \wedge s_2 \subseteq s_1)$
- $\forall x,s_1,s_2\quad x \in (s_1 \cap s_2) \Leftrightarrow (x \in s_1 \wedge x \in s_2)$
- $\forall x,s_1,s_2\quad x \in (s_1 \cup s_2) \Leftrightarrow (x \in s_1 \vee x \in s_2)$

Using FOL

The list domain (in Lisp vocabulary):

Constants: Empy list *Nil*,
Predicates: *List, ...*
Functions: *Cons, Append, First, Rest*
Syntactic sugar: []=Nil, [x|s] = *Cons*(x,s), [x], [A,B,C]

- List([])
- ∀x,l List(l) ⇔ List([x|l])
- ∀x,l First([x|l]) = x
- ∀x,l Rest([x|l]) = l
- ∀l Append([],l) = l
- ...

Example Knowledge for wumpus world

- Use list term to represent the square objects (or Square(x,y)) instead of naming each square
  ∀x,y,a,b *Adjacent*([x,y],[a,b]) ⇔
  (x=a ∧ (y=b-1 ∨ y=b+1)) ∨(y=b ∧ (x=a-1 ∨ x=a+1))
- constant *Wumpus*
- function *Home(Wumpus)* to name the one square with wumpus
- Unary Predicate *Pit(), Breezy(), Smelly()*
- Squares are breezy near a pit:

Example Knowledge for wumpus world

- Unary Predicate *Pit(), Breezy(), Smelly()*
- Squares are breezy near a pit:
  - Diagnostic rule---infer cause from effect
    $\forall s\ Breezy(s) \Leftrightarrow \exists\ r\ Adjacent(r,s) \wedge Pit(r)$
  - Causal rule---infer effect from cause
    $\forall r\ Pit(r) \Rightarrow [\forall s\ Adjacent(r,s) \Rightarrow Breezy(s)]$
    (not complete!)
    $\forall s\ [\forall r\ Adjacent(r,s) \Rightarrow \neg Pit(r)] \Rightarrow \neg\ Breezy(s)$

27

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Example Knowledge for wumpus world

- Typical percept sentence:
  Percept([Stench,Breeze,Glitter,None,None],5)
- Actions:
  Turn(Right), Turn(Left), Forward, Shoot, Grab
- Perception
  - $\forall b,g,t,m,c\ Percept([Stench,b,g,m,c],t) \Rightarrow Smell(t)$
  - $\forall s,b,t,m,c\ Percept([s,b,Glitter,m,c],t) \Rightarrow Glitter(t)$
- Properties of locations:
  $\forall s,t\ \mathit{At}(Agent,s,t) \wedge Smell(t) \Rightarrow Smelly(s)$
  $\forall s,t\ \mathit{At}(Agent,s,t) \wedge Breeze(t) \Rightarrow Breezy(s)$
- Reflex behavior
  - $\forall t\ Glitter(t) \Rightarrow BestAction(Grab,t)$

28

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Knowledge engineering in FOL

1. Identify the task (what will the KB be used for)
2. Assemble the relevant knowledge
   Knowledge acquisition.
3. Decide on a vocabulary of predicates, functions, and constants
   Translate domain-level knowledge into logic-level names.
4. Encode general knowledge about the domain
   define axioms
5. Encode a description of the specific problem instance
6. Pose queries to the inference procedure and get answers
7. Debug the knowledge base

29

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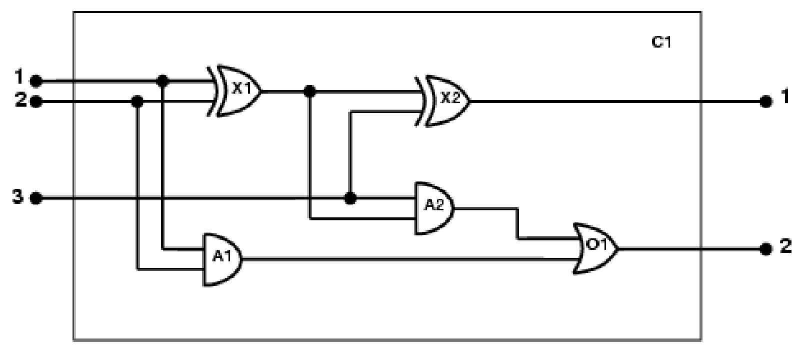
The electronic circuits domain

One-bit full adder



30

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The electronic circuits domain

1. Identify the task
   - Does the circuit actually add properly? (circuit verification)

2. Assemble the relevant knowledge
   - Composed of wires and gates;
   - Types of gates (AND, OR, XOR, NOT)
   - Connections between terminals
   - Irrelevant: size, shape, color, cost of gates

31

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The electronic circuits domain

3. Decide on a vocabulary
   - Constants: X1, X2, ..., AND, OR, ..., 1, 0
   - Functions: Type(X1), In(2,X1), Out(1,X1), Signal(Out(1,X1))
   - Alternatives:
     $Type(X_1)$ = XOR (encode that a gate can have one type)
     $Type(X_1$, XOR)
     $XOR(X_1)$
   - Predicates: Connected(Out(1,X1), In(2,X2))

32

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The electronic circuits domain

4. Encode general knowledge of the domain

- $\forall t_1,t_2\ Connected(t_1, t_2) \Rightarrow Signal(t_1) = Signal(t_2)$
- $\forall t \quad Signal(t) = 1 \vee Signal(t) = 0$
- $1 \neq 0$
- $\forall t_1,t_2\ Connected(t_1, t_2) \Rightarrow Connected(t_2, t_1)$
- $\forall g\ Type(g)=OR \Rightarrow [Signal(Out(1,g))=1 \Leftrightarrow \exists n\ Signal(In(n,g)) =1]$
- $\forall g\ Type(g)=AND \Rightarrow [Signal(Out(1,g))=0 \Leftrightarrow \exists n\ Signal(In(n,g))=0]$
- $\forall g\ Type(g) = XOR \Rightarrow [Signal(Out(1,g)) = 1 \Leftrightarrow Signal(In(1,g)) \neq Signal(In(2,g))]$
- $\forall g\ Type(g) = NOT \Rightarrow Signal(Out(1,g)) \neq Signal(In(1,g))$

The electronic circuits domain

5. Encode the specific problem instance

$Type(X_1) = XOR$ $\quad Type(X_2) = XOR$
$Type(A_1) = AND$ $\quad Type(A_2) = AND$
$Type(O_1) = OR$

$Connected(Out(1,X_1),In(1,X_2))$ $\quad Connected(In(1,C_1),In(1,X_1))$
$Connected(Out(1,X_1),In(2,A_2))$ $\quad Connected(In(1,C_1),In(1,A_1))$
$Connected(Out(1,A_2),In(1,O_1))$ $\quad Connected(In(2,C_1),In(2,X_1))$
$Connected(Out(1,A_1),In(2,O_1))$ $\quad Connected(In(2,C_1),In(2,A_1))$
$Connected(Out(1,X_2),Out(1,C_1))$ $\quad Connected(In(3,C_1),In(2,X_2))$
$Connected(Out(1,O_1),Out(2,C_1))$ $\quad Connected(In(3,C_1),In(1,A_2))$

The electronic circuits domain

6. Pose queries to the inference procedure

   What are the possible sets of values of all the terminals for the adder circuit?

   $\exists i_1,i_2,i_3,o_1,o_2$ *Signal(In(1,C_1)) =* $i_1 \wedge$ *Signal(In(2,*C_1*)) =* $i_2 \wedge$ *Signal(In(3,*C_1*)) =* $i_3 \wedge$ *Signal(Out(1,*C_1*)) =* $o_1 \wedge$ *Signal(Out(2,*C_1*)) =* o_2

7. Debug the knowledge base

   May have omitted assertions like $1 \neq 0$

35

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Summary

- First-order logic:
  - objects and relations are semantic primitives
  - syntax: constants, functions, predicates, equality, quantifiers.
- Increased expressive power: sufficient to define wumpus world

36

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