Problem 65. Prove that every infinite c.e. language has an infinite decidable subset.

Define the join of two languages $A, B \subseteq \{0, 1\}^*$ to be the language

$$A \sqcup B = \{x0 \mid x \in A\} \cup \{y1 \mid y \in B\}.$$  

Problem 66. Prove: For all $A, B \subseteq \{0, 1\}^*$,

$$A \leq_m A \sqcup B \text{ and } B \leq_m A \sqcup B.$$  

Problem 67. Prove: If $K$ is the diagonal halting problem and $K^c$ is its complement, then $K \sqcup K^c$ is neither c.e. nor co-c.e.

For each $A \subseteq \mathbb{Z}^+$, define the real number

$$x_A = \sum_{n \in A} 2^{-n}.$$  

(Assume here that our enumeration of Turing machines is such that $M_0(0) \uparrow$, so that $K \subseteq \mathbb{Z}^+$.)

Define a real number $x \in \mathbb{R}$ to be computable if there is a computable function $f_x : \mathbb{N} \to \mathbb{Q}$ such that, for all $r \in \mathbb{N}$,

$$|f_x(r) - x| \leq 2^{-r}. \quad (*)$$

Problem 68. Prove: For all $A \subseteq \mathbb{Z}^+$, $x_A$ is computable if and only if $A$ is decidable. Hence $x_K$ is an uncomputable real number.

Problem 69. Prove that there is a computable function $f : \mathbb{N} \to \mathbb{Q}$ with the following two properties:

(i) For all $t \in \mathbb{N}$, $0 \leq f(t) < x_K$.

(ii) $\lim_{t \to \infty} f(t) = x_K$. 

1
Define a *modulus* of an infinite series

\[ \sum_{n=1}^{\infty} a_n \]

of real numbers \( a_n \geq 0 \) to be a function \( m : \mathbb{N} \to \mathbb{N} \) such that, for all \( r \in \mathbb{N} \),

\[ \sum_{n=m(r)}^{\infty} a_n \leq 2^{-r}. \]

Fact: A series converges if and only if it has a modulus.

**Problem 70.** Prove: If \( g : \mathbb{Z}^+ \to \mathbb{Q} \) is computable and \( \sum_{n=1}^{\infty} g(n) \) has a computable modulus, then \( \sum_{n=1}^{\infty} g(n) \) is a computable real number.

**Problem 71.** Explain why problem 69 does not contradict problem 68.

**Problem 72.** Prove that \( \sum_{n \in K} \frac{1}{n^2} \) is not computable.