Problem 57. Prove that if languages $A$ and $B$ are computably enumerable (recursively enumerable) then the language $A \cap B$ is also computably enumerable.

Problems 58–63. Prove that the following six conditions are equivalent for a language $A \subseteq \{0,1\}^*$.

(a) $A$ is c.e.

(b) There is a computable partial function $f: \{0,1\}^* \rightarrow \{0,1\}^*$ such that $A = \text{dom } f$.

(c) There is a computable partial function $f: \{0,1\}^* \rightarrow \{0,1\}^*$ such that $A = \text{range } f$.

(d) $A = \emptyset$ or there is a computable function $f: \mathbb{N} \rightarrow \{0,1\}^*$ such that $A = \text{range } f$.

(e) $A$ is finite or there is a computable, one-to-one function $f: \mathbb{N} \rightarrow \{0,1\}^*$ such that $A = \text{range } f$.

(f) There is a decidable language $B \subseteq \{0,1\}^*$ such that

$$A = \{x \in \{0,1\}^* | (\exists w \in \{0,1\}^*) \langle x, w \rangle \in B\}.$$

(Here we are using the string-pairing function $\langle x, w \rangle = 0^{\lvert x \rvert}1xw$.)

Notes:

- In case (d) we call $f$ an enumerator of $A$, and this is the origin of the term c.e.
- In case (e) we call $f$ an enumerator of $A$ without repetition.
- We sometimes write $\exists B$ for the right-hand side of the equation in (f). A string $w$ as in (f) is called a witness that testifies that $x \in A$.

Problem 64. Prove that a language $A$ is decidable if and only if it is finite or there is a computable function $f: \mathbb{N} \rightarrow \{0,1\}^*$ such that range$(f) = A$ and each $f(n)$ comes strictly before $f(n + 1)$ in the standard enumeration of $\{0,1\}^*$.