**Problem 81.** Define the tower function $T : \mathbb{N} \to \mathbb{N}$ by the recursion

\[
T(0) = 1, \\
T(n + 1) = 2^{T(n)}.
\]

Thus $T(n)$ is a “tower” of $n$ 2’s. Prove that there are infinitely many strings $x \in \{0, 1\}^*$ that are extremely compressible in the sense that

\[
T(C(x)) < |x|.
\]

Use the KC Nonregularity Lemma to prove that the languages in problems 82 – 84 are not regular.

**Problem 82.** $\{0^n1^n \mid n \in \mathbb{N}\}$

**Problem 83.** $\{w \in \{0, 1\}^* \mid |w| \text{ is a perfect square}\}$

**Problem 84.** $\{0^m1^n \mid m, n \in \mathbb{Z}^+ \text{ and } \gcd(m, n) = 1\}$
(Note: $\gcd(m, n)$ is the greatest common divisor of $m$ and $n$.)

**Problem 85.** Use the KC Nonregularity Lemma to prove that no infinite subset of $\{0^n1^n \mid n \in \mathbb{N}\}$ is regular.

**Notation.** For $x \in \{0, 1\}^*$, let $bd(x)$ be the *bit-doubling* of $x$, i.e.,

\[
bd(\lambda) = \lambda
\]

and, for $x \in \{0, 1\}^*$ and $b \in \{0, 1\}$,

\[
bd(xb) = bd(x)bb.
\]

For $x, y \in \{0, 1\}^*$, let

\[
\langle x, y \rangle = 0^{|x|}1xy
\]
and
\[ \langle\langle x, y \rangle\rangle = bd(x)01y. \]

(Both \( \langle, \rangle \) and \( \langle\langle, \rangle\rangle \) are called \textit{string-pairing functions}.) In problems 86 – 87, let
\[
A = \{ \langle x, y \rangle \mid x, y \in \{0, 1\}^* \}, \\
B = \{ \langle\langle x, y \rangle\rangle \mid x, y \in \{0, 1\}^* \}.
\]

**Problem 86.** Prove that one of \( A \) and \( B \) is regular.

**Problem 87.** Prove that the other of \( A \) and \( B \) is not regular.

Recall that \( PAL = \{ x \in \{0, 1\}^* \mid x \text{ is a palindrome, i.e., } x^R = x \text{, where } x^R \text{ is “} x \text{ written backwards”} \} \).

Recall also that the \textit{floor} of a real number \( \alpha \) is
\[ \lfloor \alpha \rfloor = \max\{ n \in \mathbb{Z} \mid n \leq \alpha \}. \]

**Problem 88.**
(a) Prove that, for all \( n \in \mathbb{N} \),
\[ |PAL \cap \{0, 1\}^n| = 2\left\lfloor \frac{n+1}{2} \right\rfloor. \]
(b) Prove that there are infinitely many strings \( x \in PAL \) satisfying
\[ C(x) \geq \left\lfloor \frac{|x| + 1}{2} \right\rfloor. \]

(Hence the bound of problem 77 is nearly tight.)