Problem 73. Define the tower function $T : \mathbb{N} \rightarrow \mathbb{N}$ by the recursion

\[
T(0) = 1, \\
T(n + 1) = 2^{T(n)}.
\]

Thus $T(n)$ is a “tower” of $n$ 2’s. Prove that there are infinitely many strings $x \in \{0, 1\}^*$ that are extremely compressible in the sense that

\[
C(x) < T(|x|).
\]

As in class, we use the pairing function $\langle x, y \rangle = 0^{|x|}1xy$. The next two problems give upper and lower bounds on $C(\langle x, y \rangle)$.

Problem 74. Prove that there is a constant $c \in \mathbb{N}$ such that, for all $x, y \in \{0, 1\}^*$,

\[
C(\langle x, y \rangle) \leq C(x) + C(y) + \min\{C(x), C(y)\} + c.
\]

Problem 75. Prove that there is a constant $c \in \mathbb{N}$ such that, for all $x, y \in \{0, 1\}^*$,

\[
C(\langle x, y \rangle) \geq \max\{C(x), C(y)\} - c.
\]

Problem 76. Prove: For every decidable language $A \subseteq \{0, 1\}^*$ there is a constant $c \in \mathbb{N}$ such that, for all $x \in A$,

\[
C(x) \leq c + \log |A \cap \{0, 1\}^{\leq |x|}|.
\]

Let $PAL = \{x \in \{0, 1\}^* | x \text{ is a palindrome}\}$.

Problem 77. Use problem 76 to prove that there is a constant $c \in \mathbb{N}$ such that, for all $x \in PAL$,

\[
C(x) \leq \frac{|x|}{2} + c.
\]
As on page 302 of the Kozen text, let $H(x, y)$ denote the Hamming distance between strings $x, y \in \{0, 1\}^*$.

**Problem 78.** Prove that there is a constant $c \in \mathbb{N}$ such that, for all $x, y \in \{0, 1\}^+$ with $|x| = |y|$, 

$$C(y) \leq C(x) + 2 \cdot H(x, y) \cdot \log |x| + c.$$ 

Define the *characteristic sequence* of a set $A \subseteq \mathbb{Z}^+$ to be the following infinite binary sequence,

$$\chi_A = b_1 b_2 b_3 \cdots.$$ 

whose $n^{th}$ bit is

$$b_n = \begin{cases} 
1 & \text{if } n \in A \\
0 & \text{if } n \notin A.
\end{cases}$$

Write $\chi_{A,n}$ for the first $n$ bits of $\chi_A$.

**Problem 79.** Prove: For every decidable set $A \subseteq \mathbb{Z}^+$ there is a constant $c \in \mathbb{N}$ such that, for all $n \in \mathbb{Z}^+$,

$$C(\chi_{A,n}) \leq \log n + c.$$ 

**Problem 80.** Prove: For every c.e. set $A \subseteq \mathbb{Z}^+$ there is a constant $c \in \mathbb{N}$ such that, for all $n \in \mathbb{Z}^+$,

$$C(\chi_{A,n}) \leq 3 \log n + c.$$