COMS 331: Theory of Computing, Fall 2014
Homework Assignment 1

Due at the beginning class on Friday, September 5

Problem 1. Prove or disprove: If \( A = \{0^n1^n \mid n \in \mathbb{N}\} \), then \( A^* = A \).

Problem 2. Prove or disprove: If \( B = \{x \in \{0, 1\}^* \mid \#(0, x) = \#(1, x)\} \), then \( B^* = B \).

Note: The notation \( \#(0, x) \) is used to denote the number of 0’s in \( x \). Likewise, \( \#(1, x) \) is used to denote the number of 1’s in \( x \).

Problem 3. Prove: For every positive integer \( n \),

\[ \sum_{k=1}^{n} \frac{1}{k^2} \leq 2 - \frac{1}{n}. \]

Problem 4. Prove: For every language \( A \), \( A^{**} = A^* \).

Problem 5. Prove: If \( S = \{0, 1\} \) and \( T \subseteq \{0, 1\}^* \), then

\( S^* = T^* \Rightarrow S \subseteq T \).

Problem 6. Exhibit languages \( S, T \subseteq \{0, 1\}^* \) such that \( S^* = T^* \) and \( \{0, 1\} \subseteq S \subsetneq T \).

Problem 7. Define an (infinite) binary sequence \( s \in \{0, 1\}^\infty \) to be prefix-repetitive if there are infinitely many strings \( w \in \{0, 1\}^* \) such that \( ww \sqsubseteq s \).

Prove: If the bits of a string \( s \in \{0, 1\}^\infty \) are chosen by independent tosses of a fair coin, then

\[ \text{Prob}[s \text{ is prefix-repetitive}] = 0. \]

Note: \( x \sqsubseteq y \) means that \( x \) is a prefix of \( y \) where \( x \) and \( y \) are strings.
Problem 8. Define a 2-coloring of $\{0, 1\}^*$ to be a function $\chi : \{0, 1\}^* \rightarrow \{\text{red}, \text{blue}\}$. (For example, if $\chi(1101) = \text{red}$, we say that 1101 is red in the coloring $\chi$.)

Prove: For every 2-coloring $\chi$ of $\{0, 1\}^*$ and every (infinite) binary sequence $s \in \{0, 1\}^\infty$, there is a sequence

$$w_0, w_1, w_2, \cdots$$

of strings $w_n \in \{0, 1\}^*$ such that
(i) $s = w_0w_1w_2\cdots$, and
(ii) $w_1, w_2, w_3, \cdots$ are all the same color. (The string $w_0$ may or may not be this color.)