This is a closed-book, closed-notes, no-calculator, no-cellphone, individual-effort examination. All answers should be explained, at least briefly. Please do all your work on these pages.

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1. (30 points) Design a DFA that decides the language

\[ A = \{ x \in \{0,1\}^* \mid \text{num}(x) \text{ is divisible by 2 or 3 but not by 6} \} \]

(or use product construction with XOR as the Boolean function)
2. (30 points) Given $x, y \in \{0,1\}^*$, say that $x$ is a 2-deletion of $y$ if $x$ is the result of deleting at most 2 bits of $y$.

For example, the strings 1101001, 110001, and 11001 are three of the many 2-deletions of the string 1101001.

Given a language $A \subseteq \{0,1\}^*$, let

$$D_2(A) = \{ x \in \{0,1\}^* \mid \text{there exists } y \in A \text{ such that } y \text{ is a 2-deletion of } x \}\.$$ 

Prove: If $A$ is regular, then $D_2(A)$ is regular.

Let $M = (Q, \{0,1,\lambda\}, \delta, q_0, F)$ be a DFA such that $L(M) = A$. Consider the NFA with $\lambda$-transitions

$$N = (Q \times \{0,1,\lambda\}, \{0,1,\lambda\}; \Delta, (s,0), F \times \{0,1,\lambda\}),$$

where

- $\Delta((q,a), b) = \{\delta(q,b), a\lambda\}$ for all $q \in Q$, $a \in \{0,1,\lambda\}$, and $b \in \{0,1\}^*$;

- $\Delta((q,a), \lambda) = \{\delta(q,a\lambda), (\delta(q,1), a+1)\}$ for all $q \in Q$ and $a \in \{0,1\}$;

and

- $\Delta((q,\lambda), \lambda) = \emptyset$ for all $q \in Q$.

Then $L(N) = D_2(A)$, so $D_2(A)$ is regular. \(\square\)
3. (30 points) Given languages $A, B, C \subseteq \{0, 1\}^*$, let $D$ be the set of all strings $x \in \{0, 1\}^*$ with the following two properties.
   (i) $x \not\in A$.
   (ii) There exist $n \in \mathbb{N}$ and $u_1, \ldots, u_n \in \{0, 1\}^*$ such that $x = u_1 u_2 \cdots u_n$ and each $u_i \in A \cup B \cup C$.

Prove: If $A, B,$ and $C$ are regular, then $D$ is regular.

Let $\alpha, \beta,$ and $\gamma$ be regular expressions such that $L(\alpha) = A$, $L(\beta) = B$, and $L(\gamma) = C$. Then

$$D = L((\alpha + \beta \gamma)^*) - L(\alpha)$$

is the difference of two regular languages, hence is regular. $\square$
4. (30 points) Design a DFA that is equivalent to the following NFA.

Using the subset construction gives the DFA.
5. (30 points) Give a proof or counterexample for each of the following two statements.

(a) If \(A \leq \Sigma^*\) is regular and \(B \leq \Theta^\Sigma^*\) is c.e., then \(A \cap B\) is c.e.

Counterexample: If \(A = \Sigma^*\) and \(B = K\) is the diagonal halting problem, then \(A\) is regular and \(B\) is c.e., but \(A \cap B = K^c\) is not c.e.
$S$, continued.

(b) If $A \subseteq \{0,1\}^*$ is regular and $B \subseteq \{0,1\}^*$ is c.e., then $B \setminus A$ is c.e.

**Proof.** Assume the hypothesis. Then

$$B \setminus A = B \cap A^c,$$

and $A^c$ is regular, hence decidable, hence c.e., so $B \setminus A$ is the intersection of two c.e. languages, hence c.e. $\square$
6. (40 points) For each of the following conditions either

give an example of an object
of the indicated type

or

state that no such object exists.
(No proofs are required.)

(a) A pair of regular expressions \( \alpha_1 \) and \( \alpha_2 \) such that \( L(\alpha_1) \cap L(\alpha_2) \) is not denoted by any regular expression.

No such object.

(b) A nonregular language \( B \subseteq \{0,1\}^* \) such that \( |B \cap \{0,1\}^n| \geq 2^n - 1 \) for all \( n \in \mathbb{N} \).

\[ B = \{0,1\}^* \setminus \{0^n 1^n \mid n \in \mathbb{N} \} \]
6. continued.

(e) A language $C$ that is c.e. and co-c.e. but not regular.

$$C = \{0^n1^n \mid n \in \mathbb{N} \}.$$  

(d) A language $D$ that is co-c.e. but not decidable.

$$D = K^c.$$  

(e) A pair of languages $E_1, E_2 \subseteq \{0,1\}^*$ such that $E_1 \leq_m E_2$ and $E_2 \leq_m E_1$ but $E_1 \neq E_2$.

$$E_1 = \{0^n1^n \mid n \in \mathbb{N} \}, \quad E_2 = \{1^n0^n \mid n \in \mathbb{N} \}.$$
(f) A pair of languages $F_1$ and $F_2$ such that $F_1 \subseteq F_2$ and $F_2$ is decidable, but $F_1$ is not r.e.

$$F_1 = K^c, \quad F_2 = \{0, 1\}^*.$$ 

(g) A natural number $n$ such that $C(x) < n$ holds for all $x \in \{0, 1\}^n$.

No such $n$ exists.

(h) Two uncomputable real numbers $x$ and $y$ whose sum is computable.

$$x = \sum_{n \in \mathbb{N}} 2^{-n}, \quad y = -x.$$
7. (40 points) Define \( h : \mathbb{Z}^+ \rightarrow \mathbb{Z}^+ \) by
\[
h(n) = \begin{cases} 
\frac{n}{2} & \text{if } n \text{ is even} \\
3n+1 & \text{if } n \text{ is odd}
\end{cases}
\]
For each \( k \in \mathbb{N} \) define \( h^k : \mathbb{Z}^+ \rightarrow \mathbb{Z}^+ \) by the recursion
\[
h^0(n) = n, \\
h^{k+1}(n) = h(h^k(n)).
\]
Prove that the set
\[
A = \{ n \in \mathbb{Z}^+ \mid \text{there exists } k \in \mathbb{N} \text{ such that } h^k(n) = 1 \}
\]
is computably enumerable.

Let \( B = \{ (n, k) \mid h^k(n) = 1 \} \). Then \( B \) is decidable and \( A = \exists B \), so \( B \) is c.e.
Name  KEY
8. (40 points) Let \[ x = \sum_{n=1}^{\infty} 2^{-n^2} \]
be the real number whose binary expansion is
\[ x = 0.100100001000000100\ldots \]
(This expansion is nonrepeating, so \( x \) is an irrational number.)
Prove that \( x \) is a computable real number.

Define \( f: \mathbb{N} \rightarrow \mathbb{Q} \) by
\[ f(r) = \sum_{n=1}^{r} 2^{-n^2} \]
for all \( r \in \mathbb{N} \). Then \( f \) is computable. For all \( r \in \mathbb{N} \) we have
\[ 0 \leq x - f(r) = \sum_{n=r+1}^{\infty} 2^{-n^2} < \sum_{n=r+1}^{\infty} 2^{-r} = 2^{-r} \]
so \( |f(r) - x| < 2^{-r} \). Hence \( f \) testifies that \( x \) is computable. \[ \square \]
Name ____________________

KEY
9. (40 points) Let \( n \) be a positive integer, and let \( A \subseteq \{0,1\}^{3n} \) satisfy \(|A| \leq 2^n\). Prove that there is a constant \( c \in \mathbb{N} \) such that, for all \( x \in A^* \),

\[
C(x) \leq \frac{|x|}{3} + c.
\]

Assume the hypothesis. Since \(|A| \leq 2^n\) there is a function \( f : \{0,1\}^n \rightarrow A \). Let \( M \) be a TM such that, for all \( m \in \mathbb{N} \) and \( \pi_1, \ldots, \pi_m \in \{0,1\}^n \),

\[
M(\pi_1, \ldots, \pi_m) = f(\pi_1) \cdots f(\pi_m),
\]

and let \( c = cm \) be an optimality constant for \( M \).

To see that \( c \) has the desired property, let \( x \in A^* \). Then there exist \( u_1, \ldots, u_m \in A \) such that \( x = u_1 \cdots u_m \). Since \( f \) is onto, there exist \( \pi_1, \ldots, \pi_m \in \{0,1\}^n \) such that each \( f(\pi_i) = u_i \).

Then

\[
M(\pi_1, \ldots, \pi_m) = f(\pi_1) \cdots f(\pi_m) = u_1 \cdots u_m = x,
\]

so

\[
C(x) \leq C_M(x) + c \leq |\pi_1, \ldots, \pi_m| + c = mn + c = \frac{3mn}{3} + c = \frac{|x|}{3} + c.
\]
10. (40 points) For each $n \in \mathbb{N}$ define the string $x_n \in \{0,1\}^{2n}$ by the recursion

$$
\begin{align*}
    x_0 &= \lambda, \\
    x_{2n+1} &= x_{2n} 1^{2n+1}, \\
    x_{2n+2} &= x_{2n+1} 0^{2n+2}.
\end{align*}
$$

Thus, for example,

$$x_5 = 1001110000011111.$$

Use the KC nonregularity theorem to prove that the language

$$A = \{ x_n \mid n \in \mathbb{N} \}$$

is not regular.

Let $d \in \mathbb{N}$. Choose $k \in \mathbb{N}$ such that

$$C(12^{k+1}) > d+1.$$  Let $x = x_{2k}$. Then

$$C(\sqrt{x}) = C(12^{k+1}) > d+1 = d + \log 2.$$
Name: KEY