Parametric Curves

Com S 229

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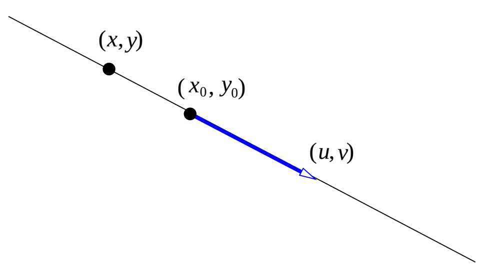
1 Introduction

A *curve* in $\mathbb{R}^2$ is a differentiable function $\boldsymbol{\alpha} : [a, b] \to \mathbb{R}^2$. The *initial point* is $\boldsymbol{\alpha}[a]$ and the *final point* is $\boldsymbol{\alpha}[b]$. The *domain* of the curve is the interval $[a, b]$. A portion of $\boldsymbol{\alpha}$ defined on an interval $[c, d] \subseteq [a, b]$ is called a *curve segment*.

Example 1. **Straight Line** The line is the simplest curve in the plane as its coordinate functions are linear. Explicitly, the curve

$$\boldsymbol{\alpha}(t) = \boldsymbol{p} + t\boldsymbol{v} = (x_0 + tu, y_0 + tv), \qquad \text{where } \boldsymbol{v} \neq \mathbf{0}, \tag{1}$$

is a straight line through the reference point $\boldsymbol{p} = \boldsymbol{\alpha}(0) = (x_0, y_0)$ in the direction $\boldsymbol{v} = (u, v)$. Here, t is the signed distance from a point $\boldsymbol{\alpha}(t)$ on the line to $\boldsymbol{p}$ as scaled by $\|\boldsymbol{v}\|$.



As shown on the left, the vector from $\boldsymbol{p}$ to a point (x, y) on the line must be either in the direction of (u, v) or in its opposite direction. Hence, the cross product of the two vectors must be zero, that is,

$$(x - x_0, y - y_0) \times (u, v) = 0.$$

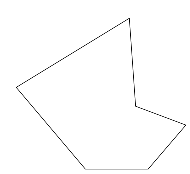
Expansion of the above cross product yields an *implicit equation* of the line that relates the x and y coordinates of every incident point:

$$vx - uy - vx_0 + uy_0 = 0. \tag{2}$$

A curve $\boldsymbol{\alpha}(t) = (x(t), y(t))$ is said to be *smooth* at $t = t_0$ if its kth derivative

$$\boldsymbol{\alpha}^{(k)}(t) = \left(x^{(k)}(t), y^{(k)}(t)\right)$$

exists for any integer $k > 0$. A *piecewise smooth curve* $\boldsymbol{\alpha}$ has a domain which is the union of a finite number of subintervals over each of which $\boldsymbol{\alpha}$ is smooth.



Example 2. A line $\boldsymbol{\alpha}(t) = \boldsymbol{p} + t\boldsymbol{v}$ is a smooth curve, since $\boldsymbol{\alpha}'(t) = \boldsymbol{v}$ and $\boldsymbol{\alpha}^{(k)} = \mathbf{0}$ for $k > 1$. A polygon, on the other hand, is a piecewise smooth curve, where each edge determines a subdomain.

Example 3. **Crunodal cubic** is described by the parametric equation

$$\boldsymbol{\alpha}(t) = (c(t^2 - 1), dt(t^2 - 1)). \tag{3}$$

It is smooth, as plotted in the next figure for the case $c = d = 1$.[1]

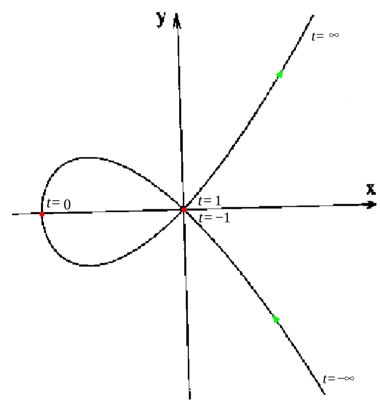
[1] This figure is taken from [2, p. 60].

We obtain

$$\begin{aligned}
\boldsymbol{\alpha}'(t) &= (2ct, 3dt^2 - d), \\
\boldsymbol{\alpha}''(t) &= (2c, 6dt), \\
\boldsymbol{\alpha}'''(t) &= (0, 6d), \\
\boldsymbol{\alpha}^{(k)}(t) &= \mathbf{0}, \qquad k \geq 4.
\end{aligned}$$

The curve also has an implicit equation

$$\frac{y^2}{d^2} = \frac{x^2}{c^2}\left(\frac{x}{c} + 1\right). \tag{4}$$



Consider a plane curve $\boldsymbol{\alpha} : [a, b] \to \mathbb{R}^2$. It is called a *closed parametric curve* if $\boldsymbol{\alpha}(a) = \boldsymbol{\alpha}(b)$. A point of *self-crossing* is a point $\boldsymbol{\alpha}(t_1)$ for which there exist finitely many distinct values $t_1, \ldots, t_n \in [a, b]$, $n \geq 2$, which satisfy $\boldsymbol{\alpha}(t_1) = \boldsymbol{\alpha}(t_2) = \cdots = \boldsymbol{\alpha}(t_n)$, and in the case $n = 2$, $[t_1, t_2] \neq [a, b]$. If α does have a self-crossing point, it is called a *simple curve.*

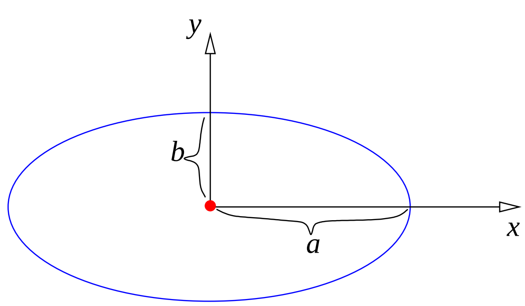
Example 4. **Ellipse** An ellipse is closed, with the parametrization

$$\boldsymbol{\alpha}(\theta) = (a\cos\theta, b\cos\theta), \qquad \theta \in [0, 2\pi), \tag{5}$$

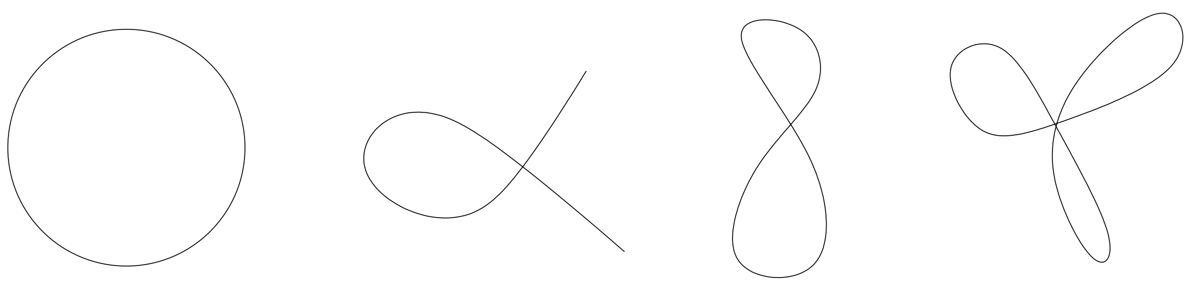
where $a, b > 0$. The ellipse also has the implicit equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1. \tag{6}$$

When $a = b$, it becomes a circle with radius a.



The circle shown in the next figure is closed, while the other three curves all have self-crossings.



2 Velocity, Speed, and Arc Length

The *velocity vector* of a curve $\boldsymbol{\alpha}(t)$ at t is $\boldsymbol{\alpha}'(t)$. Its *speed* v at t is the length $\|\boldsymbol{\alpha}'(t)\|$. The meaning is clear if we see $\boldsymbol{\alpha}(t)$ as the location of a moving point at time t. The parametrization $\boldsymbol{\alpha}(t)$ is *unit-speed* if $\|\boldsymbol{\alpha}'(t)\| = 1$ for all values of t. A point where $\boldsymbol{\alpha}'(t) = \mathbf{0}$ is called a *cusp* on the curve. The curve $\boldsymbol{\alpha}(t)$ is *regular* if all velocity vectors are different from zero, that is, $\boldsymbol{\alpha}'(t) \neq \mathbf{0}$ for all t. So the moving point never loses its velocity on a regular curve, let alone reverses its motion.

Example 5. Consider the curve $\boldsymbol{\alpha}(\theta) = (a\theta\cos\theta, a\theta\sin\theta)$. It has velocity

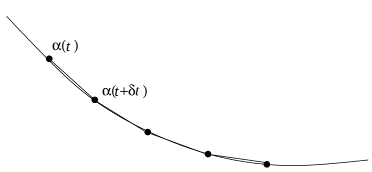
$$\boldsymbol{\alpha}'(\theta) = a(\cos\theta - \theta\sin\theta, \sin\theta + \theta\cos\theta),$$

and speed

$$\|\boldsymbol{\alpha}'(\theta)\| = |a|\sqrt{(\cos\theta - \theta\sin\theta)^2 + (\sin\theta + \theta\cos\theta)^2} = |a|\sqrt{1+\theta^2} \neq 0.$$

Therefore, the parametrization is regular.

The velocity and speed depend on the parametrization. Generally, they vary with the parametrization of the same geometric curve. Non-regularity at a point may be just a property of the parametrization, and need not correspond to any special feature of the geometric curve. For a different parametrization the curve may have a non-zero velocity at the corresponding point.



To formulate the length of $\boldsymbol{\alpha}$, we note that the portion over $[t, t+\delta t]$ is nearly a straight line when δt is very small. So the length over $[t, t+\delta t]$ can be approximated by

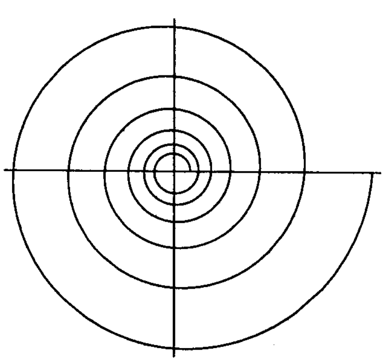
$$\|\boldsymbol{\alpha}(t+\delta t) - \boldsymbol{\alpha}(t)\|,$$

which again is approximated by

$$\|\boldsymbol{\alpha}'(t)\|\delta t.$$

To calculate the length of a segment of $\boldsymbol{\alpha}$, we divide it up into segments, each of which corresponds to a small increment δt. As δt tends to zero, we will obtain the exact length.

The *arc length* of $\boldsymbol{\alpha}$ from $t = a$ to $t = b$ is thus defined as



$$\int_a^b \|\boldsymbol{\alpha}'(t)\|\, dt. \tag{7}$$

If $\boldsymbol{\alpha}'(t)$ is a unit-speed curve, i.e., $\|\boldsymbol{\alpha}'(t)\| = 1$, the arc length over $[a, b]$ is simply $b - a$. In this case, suppose $t = t_0$ corresponds to the starting point on the curve, then $t - t_0$ simply gives the arc length from the starting point. In other words, *a unit-speed curve is parametrized with arc length.*

Example 6. **Logarithmic spiral**[2]

The curve

$$\boldsymbol{\alpha}(t) = (e^t\cos t, e^t\sin t),$$

[2]The figure originally appears in [3, p. 8].

exhibits a spiral motion. We obtain that

$$\begin{aligned}\boldsymbol{\alpha}'(t) &= \left(e^t(\cos t - \sin t), e^t(\sin t + \cos t)\right), \\ \|\boldsymbol{\alpha}'(t)\| &= \sqrt{2}e^t.\end{aligned}$$

Hence the arc length of $\boldsymbol{\alpha}$ starting at $\boldsymbol{\alpha}(0) = (1, 0)$, for instance, is

$$s = \int_0^t \sqrt{2}e^u \, du = \sqrt{2}(e^t - 1).$$

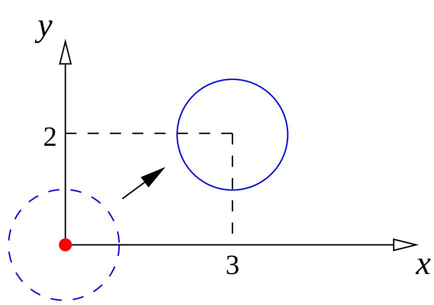
2.1 Numerical Computation of Arc Length

In case no closed form for the integral (7) exists, the arc length needs to be evaluated numerically. We discretize the domain $[a, b]$ into n intervals of equal length $h = \frac{b-a}{n}$. Then the arc length integral can be approximated using the *trapezoidal rule* on integration:

$$\begin{aligned}\int_a^b \|\boldsymbol{\alpha}'(t)\| \, dt &\approx \sum_{i=1}^{n} \frac{h}{2}\Big(\|\boldsymbol{\alpha}'(a + (i-1)h)\| + \|\boldsymbol{\alpha}'(a + ih)\|\Big) \\ &= \frac{h}{2}\Big(\|\boldsymbol{\alpha}'(a)\| + \|\boldsymbol{\alpha}'(b)\|\Big) + h\sum_{i=1}^{n-1} \|\boldsymbol{\alpha}'(a + ih)\|. \qquad (8)\end{aligned}$$

The smaller value h has, the better the accuracy.

3 Translation of a Curve



Translate a curve $\boldsymbol{\alpha}(t)$ by (c, d). Under the motion, every point $(x(t), y(t))$ on the curve moves to $(x(t) + c, y(t) + d)$, yielding a new curve $\beta(t)$. Suppose the original curve $\boldsymbol{\alpha}(t)$ is described by the implicit equation $f(x, y) = 0$. Then the curve $\beta(t)$ satisfies the implicit equation

$$g(x, y) \equiv f(x - c, y - d) = 0. \qquad (9)$$

Example 7. Translates the unit circle $x^2 + y^2 = 1$ so that its center moves from the origin to $(3, 2)$. The new circle has the equation

$$(x - 3)^2 + (y - 2)^2 = 1.$$

4 Intersection of Two Curves

Two curves can intersect at more than one points. Locating the intersection points is essentially a matter of root finding. Suppose one curve is parametrized as $\boldsymbol{\alpha}(t) = (x(t), y(t))$ over the interval

$[a, b]$, and the other curve β is described by an implicit equation $f(x, y) = 0$. Then we substitute the parametrization $(x(t), y(t))$ into the implicit equation, obtaining an equation in one variable t:

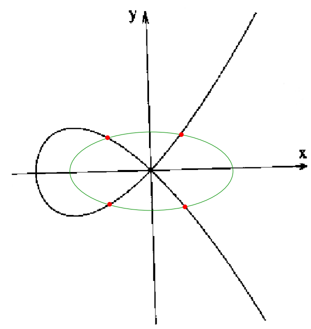
$$f(x(t), y(t)) = 0.$$

Find all the roots of the above equation. Every real root $t = r \in [a, b]$ then corresponds to an intersection point $(x(r), y(r))$.

EXAMPLE 8. Let us intersect an ellipse and a crunodal cubic. We use the implicit equation (6) of the ellipse and the parametrization (3),

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \qquad \text{and} \qquad (c(t^2 - 1), dt(t^2 - 1)).$$

Every intersection point lies on both the ellipse and the crunodal cubic, hence satisfying both the implicit equation and the parametrization above. We substitute the parametrization into the implicit equation, deriving a sequence of equivalent equations in t:



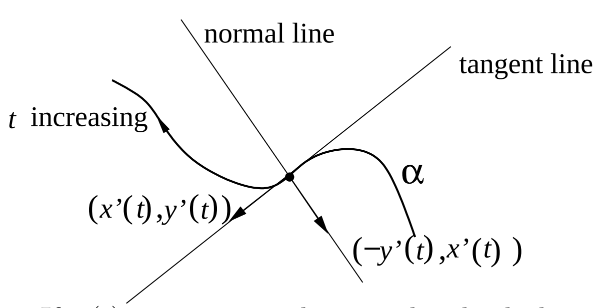
$$\begin{aligned} \frac{c^2}{a^2}(t^2-1)^2 + \frac{d^2}{b^2}t^2(t^2-1)^2 &= 1, \\ \frac{c^2}{a^2}(t^4-2t^2+1) + \frac{d^2}{b^2}t^2(t^4-2t^2+1) &= 1, \\ \frac{d^2}{b^2}t^6 + \left(\frac{c^2}{a^2} - \frac{2d^2}{b^2}\right)t^4 + \left(\frac{d^2}{b^2} - \frac{2c^2}{a^2}\right)t^2 + \frac{c^2}{a^2} - 1 &= 0. \qquad (10) \end{aligned}$$

Equation (10) is a polynomial of degree 6 whose real roots must come in pairs of values of opposite signs. We can either solve it directly, or substitute $s = t^2$ and solve the resulting cubic polynomial in terms of s. Since the parameter t ranges from $-\infty$ to ∞, every real root corresponds to an actual intersection point.

In the above example, both the ellipse and the crunodal cubic have no translations. If one of them has been translated, the implicit equation or the parametrization can be derived as shown in Section 3. The resulting polynomial equation in t is left as an exercise.

5 Tangent and Normal

The standard method of studying the geometry of a curve at a point is to attach orthonormal vectors to the point and see how the directions of these vectors change as the point moves on the curve for an infinitesimal distance. We choose tangent and normal vectors at a regular point.



Let $\boldsymbol{\alpha}(t) = (x(t), y(t))$ be a curve. At a regular point $\boldsymbol{\alpha}(t)$ we have a (non-zero) *tangent vector* $\boldsymbol{\alpha}'(t) = (x'(t), y'(t))$. So the tangent vector represents the velocity of the curve at the point. The normal vector at $\boldsymbol{\alpha}(t)$ is given by rotating the tangent vector counterclockwise through an angle $\frac{\pi}{2}$. It is given by $(-y'(t), x'(t))$. Note that $(x'(t), y'(t)) \times (-y'(t), x'(t)) = (x'(t))^2 + (y'(t))^2 > 0$.

If $\boldsymbol{\alpha}(t)$ is a unit-speed curve, then both the tangent vector and the normal vector are unit vectors. By convention they are denoted as T and N, respectively, with the cross product $T \times N = 1$.

For a parametric curve we have a tangent line and a normal line passing through each regular point $\boldsymbol{\alpha}(t)$. The tangent line to the curve at $\boldsymbol{\alpha}(t)$ passes through $\boldsymbol{\alpha}(t)$ and is parallel to $\boldsymbol{\alpha}'(t) \neq 0$. So it has the parametric equation

$$\Big(x(s), y(s)\Big) = \boldsymbol{\alpha}(t) + s\boldsymbol{\alpha}'(t), \qquad s \in (-\infty, \infty),$$

or equivalently, the algebraic equation

$$\Big((x, y) - \boldsymbol{\alpha}(t)\Big) \cdot \Big(-y'(t), x'(t)\Big) = 0.$$

The normal line at $\boldsymbol{\alpha}(t)$ passes through the point and is parallel to $(-y'(t), x'(t))$. So its equations are of the form

$$\Big(x(s), y(s)\Big) = \boldsymbol{\alpha}(t) + s\Big(-y'(t), x'(t)\Big), \qquad s \in (-\infty, \infty),$$

or equivalently,

$$\bigg(\Big(x(s), y(s)\Big) - \boldsymbol{\alpha}(t)\bigg) \cdot \boldsymbol{\alpha}'(t) = 0.$$

Example 8. Find the tangent and normal lines of the crunodal cubic (cf. Example 3):

$$\boldsymbol{\alpha}(t) = \Big(t^2 - 1, t(t^2 - 1)\Big)$$

at the points $t = \pm 1, 0$.

We obtain that

$$\begin{aligned} \boldsymbol{\alpha}'(t) &= (2t, 3t^2 - 1), \\ \boldsymbol{\alpha}'(1) &= (2, 2), \\ \boldsymbol{\alpha}'(-1) &= (-2, 2), \\ \boldsymbol{\alpha}'(0) &= (0, -1), \\ \boldsymbol{\alpha}(\pm 1) &= (0, 0). \end{aligned}$$

Here $\boldsymbol{\alpha} = (0, 0)$ is referred to as a *double point* since it is attained at both $t = 1$ and $t = -1$. The tangent lines at this double point are respectively

$$(x, y) = s(1, 1), \qquad \text{or equivalently,} \qquad y = x,$$

and

$$(x, y) = s(-1, 1), \qquad \text{or equivalently,} \qquad y = -x.$$

The normal lines at the double point are respectively

$$(x, y) = s(-1, 1), \qquad \text{or equivalently,} \qquad y = -x,$$

and

$$(x, y) = s(-1, -1), \qquad \text{or equivalently,} \qquad y = x.$$

At $t = 0$, we have $\boldsymbol{\alpha}'(0) = (0, -1)$, and the tangent line at $\boldsymbol{\alpha}(0)$ is

$$(x, y) = (-1, 0) + s(0, -1), \qquad \text{or equivalently,} \qquad x = -1.$$

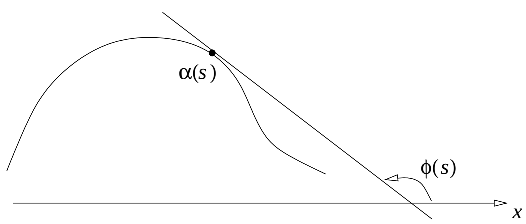
The normal line at $\boldsymbol{\alpha}(0)$ is

$$(x, y) = (-1, 0) + s(1, 0), \qquad \text{or equivalently,} \qquad y = 0.$$

6 Curvature

We want to find a measure of how 'curved' a curve is. This "curvature" should depend only on the 'shape' of the curve. It should not be changed when the curve is reparametrized. Further, the measure of curvature should agree with our intuition in simple special cases. Straight lines themselves have zero curvature. Large circles should have smaller curvature than small circles which bend more sharply.

The (signed) curvature of a curve parametrized by its arc length is the rate of change of direction of the tangent vector. The absolute value of the curvature is a measure of how sharply the curve bends. Curves which bend slowly, which are almost straight lines, will have small absolute curvature. Curves which swing to the left have positive curvature and curves which swing to the right have negative curvature. The curvature of the direction of a road will affect the maximum speed at which vehicles can travel without skidding, and the curvature in the trajectory of an aeroplane will affect whether the pilot will suffer "blackout" as a result of the g-forces involved.



To introduce the definition of curvature, in this section we consider that $\boldsymbol{\alpha}(s)$ *is a unit-speed curve*, so s is the arc length. The *tangential angle* ϕ is measured counterclockwise from the x-axis to the unit tangent $T = \boldsymbol{\alpha}'(s)$, as shown on the left.

The *curvature* κ of $\boldsymbol{\alpha}$ is the rate of change of direction at that point of the tangent line with respect to arc length, that is,

$$\kappa = \frac{d\phi}{ds}. \tag{11}$$

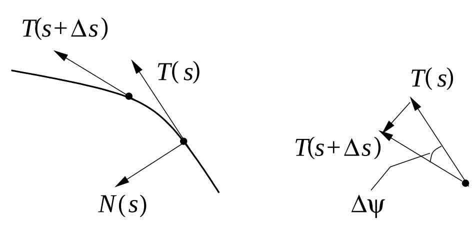
Since $\boldsymbol{\alpha}$ has unit speed, $T \cdot T = 1$. Differentiating this equation yields

$$T' \cdot T = 0.$$

The change of $T(s)$ is orthogonal to the tangential direction, so it must be along the normal direction or its opposite direction. The curvature is also defined to measure the turning of $T(s)$ along the direction of the unit normal $N(s)$. That is,

$$T' = \frac{dT}{ds} = \kappa N. \tag{12}$$

We can easily derive one of the curvature definitions (10) and (12) from the other. For instance, if we start with (12), then



$$\begin{aligned}
\kappa &= T' \cdot N \\
&= \frac{dT}{ds} \cdot N \\
&= \lim_{\Delta s \to 0} \frac{T(s+\Delta s) - T(s)}{\Delta s} \cdot N \\
&= \lim_{\Delta s \to 0} \frac{\Delta\phi \cdot \|T\|}{\Delta s}
\end{aligned}$$

$$= \lim_{\Delta s \to 0} \frac{\Delta \phi}{\Delta s}$$
$$= \frac{d\phi}{ds}.$$

EXAMPLE 9. Let us compute the curvature of the unit-speed circle

$$\boldsymbol{\alpha}(s) = r\left(\cos\frac{s}{r}, \sin\frac{s}{r}\right).$$

We obtain that

$$\begin{aligned} T &= \boldsymbol{\alpha}'(s) = \left(-\sin\frac{s}{r}, \cos\frac{s}{r}\right), \\ N &= \left(-\cos\frac{s}{r}, -\sin\frac{s}{r}\right), \\ T' &= \boldsymbol{\alpha}''(s) = -\frac{1}{r}\left(\cos\frac{s}{r}, \sin\frac{s}{r}\right) = \frac{1}{r}N. \end{aligned}$$

Thus

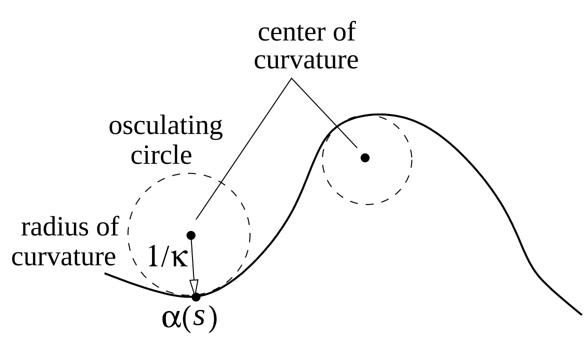
$$\kappa(s) = \frac{1}{r}. \qquad \text{cf. (12)}$$

The curvature of a circle equals the inverse of its radius everywhere.

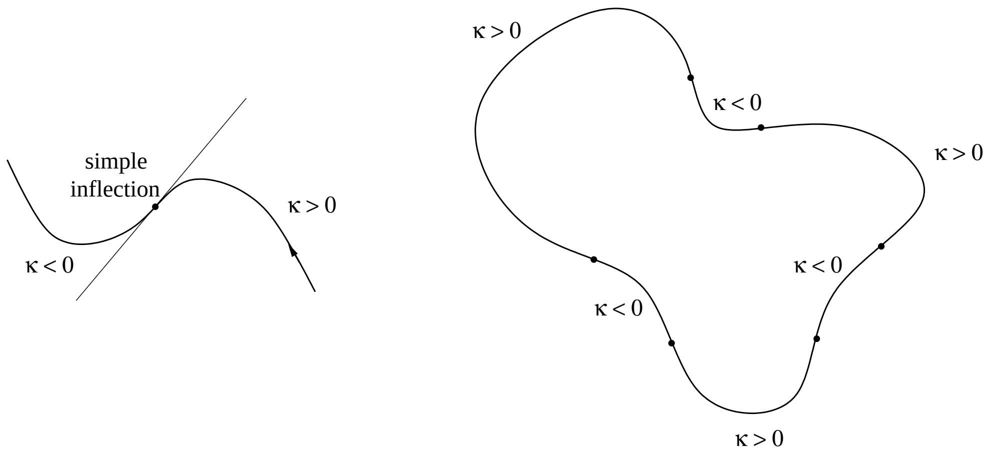
When the curvature $\kappa(s) > 0$, the *center of curvature* lies along the direction of $N(s)$ at distance $\frac{1}{\kappa}$ from the point $\boldsymbol{\alpha}(s)$. When $\kappa(s) < 0$, the center of curvature lies along the direction of $-N(s)$ at distance $-\frac{1}{\kappa}$ from $\boldsymbol{\alpha}(s)$. In either case, the center of curvature is located at

$$\boldsymbol{\alpha}(s) + \frac{1}{\kappa(s)} N(s).$$

The *osculating circle*, when $\kappa \neq 0$, is the circle at the center of curvature with radius $\frac{1}{|\kappa|}$, which is called the *radius of curvature*. The osculating circle approximates the curve locally up to the second order.



A point s on the curve $\boldsymbol{\alpha}$ is *simple inflection*, or *inflection*, if the curvature $\kappa(s) = 0$ but $\kappa'(s) \neq 0$. Intuitively, a simple inflection is where the curve swing from the left of the tangent at the point to its right; or in the case of simple closed curve, it is where the closed curve $\boldsymbol{\alpha}$ changes from convex to concave or from concave to convex. In the figure below, the curve on the left has one simple inflection while the curve on the right has six simple inflections.



Every point on a line has zero curvature, that is, $\kappa \equiv 0$. Since κ' also vanishes everywhere, a line does not have an inflection.

7 Curvature of Arbitrary-Speed Curves

Let $\boldsymbol{\alpha}(t)$ be a regular curve but not necessarily unit-speed. We obtain the unit tangent as $T = \boldsymbol{\alpha}'/\|\boldsymbol{\alpha}'\|$ and the unit normal N as the counterclockwise rotation of T by $\frac{\pi}{2}$. Still denote by v and $\kappa(t)$ the speed and curvature functions, respectively. Here $v = \|\boldsymbol{\alpha}'\|$. It can be shown that the derivative of T with respect to t is

$$T' = \kappa v N. \tag{13}$$

Now let $\boldsymbol{\alpha}(t) = (x(t), y(t))$. Then

$$\begin{aligned} T &= (x', y')/\|\boldsymbol{\alpha}'(t)\| = (x', y')\Big/\sqrt{x'^2 + y'^2}, \\ N &= (-y', x')\Big/\sqrt{x'^2 + y'^2}. \end{aligned}$$

Substituting these terms into (13) yields a formula for curvature evaluation:

$$\begin{aligned} \kappa &= \frac{T' \cdot N}{v} \\ &= \left(\frac{(x'', y'')}{\sqrt{x'^2 + y'^2}} + \frac{d}{dt}\left(\frac{1}{\sqrt{x'^2 + y'^2}} \right)(x', y') \right) \cdot \frac{(-y', x')}{\sqrt{x'^2 + y'^2}} \Big/ \sqrt{x'^2 + y'^2} \\ &= \frac{x'y'' - x''y'}{(x'^2 + y'^2)^{\frac{3}{2}}} + \frac{d}{dt}\left(\frac{1}{\sqrt{x'^2 + y'^2}} \right) \cdot \frac{(x', y') \cdot (-y', x')}{x'^2 + y'^2} \\ &= \frac{x'y'' - x''y'}{(x'^2 + y'^2)^{\frac{3}{2}}}. \end{aligned} \tag{14}$$

We can write the formula simply as

$$\kappa = \frac{\boldsymbol{\alpha}' \times \boldsymbol{\alpha}''}{\|\boldsymbol{\alpha}'\|^3}.$$

EXAMPLE 10. Find the curvature of the curve $\boldsymbol{\alpha}(t) = (t^3 - t, t^2)$. so we have

$$\begin{aligned}\boldsymbol{\alpha}'(t) &= (3t^2 - 1, 2t),\\ \boldsymbol{\alpha}''(t) &= (6t, 2).\end{aligned}$$

Therefore

$$\begin{aligned}\kappa &= \frac{x'y'' - x''y'}{(x'^2 + y'^2)^{\frac{3}{2}}}\\ &= \frac{(3t^2 - 1) \cdot 2 - 2t \cdot 6t}{\left((3t^2 - 1)^2 + (2t)^2\right)^{\frac{3}{2}}}\\ &= -\frac{6t^2 + 2}{(9t^4 - 2t^2 + 1)^{\frac{3}{2}}}.\end{aligned}$$

Simple inflections on a regular curve can be found by determining the roots of the equation

$$x'y'' - x''y' = 0.$$

Every real root r in the curve domain $[a, b]$ corresponds to an inflection $(x(r), y(r))$. In the case both $x(t)$ and $y(t)$ are polynomials, the above equation can be solved using Müller's method or closed forms when the degree does not exceed 4.

References

[1] B. O'Neill. *Elementary Differential Geometry.* Academic Press, Inc., 1966.

[2] J. W. Rutter. *Geometry of Curves.* Chapman & Hall/CRC, 2000.

[3] A. Pressley. *Elementary Differential Geometry.* Springer-Verlag London, 2001.