# Chapter 12

# Introduction to CTMCs

We will assume that $\mathcal{T} = \mathbb{R}^* = [0, \infty)$, the non-negative reals.

**Property 12.1** *CTMC property:*

$$\Pr\{X(t) = x | X(t_n) = x_n, \ldots, X(t_0) = x_0\} = \Pr\{X(t) = x | X(t_n) = x_n\}$$

*where $t > t_n > \cdots > t_0$ are all times in $\mathcal{T}$, and $x, x_n, \ldots, x_0$ are all states in $\mathcal{S}$.*

Again, we will only consider *homogeneous* CTMCs, where

$$\Pr\{X(t+h) = j | X(t) = i\} = \Pr\{X(h) = j | X(0) = i\}$$

i.e., the probabilities depend on the *time difference*, not the actual time. For shorthand, we will denote this as a matrix

$$\mathbf{P}_h[i,j] = \Pr\{X(h) = j | X(0) = i\}$$

For now, we will assume that *transitions between states are not arbitrarily fast*[1]:

$$\lim_{h \to 0} \mathbf{P}_h = \mathbf{I}$$

I.e., as the time difference goes to zero, the probability of changing states goes to zero.

In the discrete case, we discussed how the process evolves with every "clock tick", i.e., as time advances by discrete steps. Since time is now continuous, to discuss how the process evolves at an infinitesimally–small time instant, we must take the derivative of $\mathbf{P}_h$. Since we are considering the homogeneous case, this derivative must be independent of the time instant, so we will look at time zero. In particular, let $\mathbf{Q}$ be the derivative of $\mathbf{P}_h$ at time 0:

$$\begin{aligned} \mathbf{Q} &= \lim_{h \to 0} \frac{\mathbf{P}_h - \mathbf{P}_0}{h} \\ &= \lim_{h \to 0} \frac{\mathbf{P}_h - \mathbf{I}}{h} \end{aligned}$$

The units of each element $\mathbf{Q}[i,j]$ are "probability per time unit"; or, think of it as a "flow of probability from state $i$ to state $j$"; or, think of it as "how does the probability of going from state $i$ to state $j$ increase over time".

Let's examine the entries of matrix $\mathbf{Q}$.

[1] We will discuss how to relax this restriction later.

- The "off–diagonal" elements:
$$\mathbf{Q}[i,j] = \lim_{h\to 0} \frac{\mathbf{P}_h[i,j] - 0}{h}$$
Since $\mathbf{P}_h[i,j]$ is a probability, and $\mathbf{P}_0[i,j]$ is zero, $\mathbf{P}_h[i,j]$ can only increase (or remain the same), so we must have $\mathbf{Q}[i,j] \geq 0$.

- The "diagonal" elements:
$$\mathbf{Q}[i,i] = \lim_{h\to 0} \frac{\mathbf{P}_h[i,i] - 1}{h}$$
Again, since $\mathbf{P}_h[i,i]$ is a probability, and $\mathbf{P}_0[i,i]$ is one, $\mathbf{P}_h[i,i]$ can only decrease (or remain the same), so we must have $\mathbf{Q}[i,i] \leq 0$.

For any time $t$, though, we must have that the rows of $\mathbf{P}_t$ sum to one. Therefore the rows of $\mathbf{P}_t - \mathbf{I}$ sum to zero. Thus, the rows of $\mathbf{Q}$ sum to zero. Therefore, we must have that

**Property 12.2**

$$\mathbf{Q}[i,i] = -\sum_{j\neq i} \mathbf{Q}[i,j]$$

*for all $i \in \mathcal{S}$.*

This property makes sense: if the probability of going to some other state from state $i$ increases with time, then the probability of remaining in state $i$ must decrease with time, at the same rate.

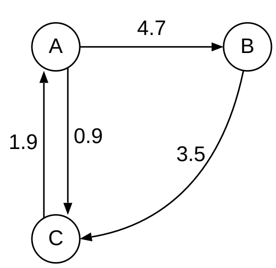
**Definition 12.3** *The matrix*
$$\mathbf{Q} = \lim_{h\to 0} \frac{\mathbf{P}_h - \mathbf{I}}{h}$$
*is called the infinitesimal generator matrix.*

Sometimes we use the *transition rate matrix* $\mathbf{R}$:

$$\mathbf{R}[i,j] = \begin{cases} \mathbf{Q}[i,j] & \text{if } i \neq j \\ 0 & \text{if } i = j \end{cases}$$

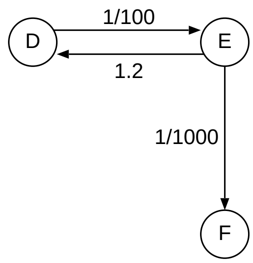
CTMCs are usually drawn as weighted, directed graphs with incidence matrix $\mathbf{R}$.

**Example 12.1**




$$\mathbf{R} = \begin{bmatrix} 0 & 4.7 & 0.9 \\ 0 & 0 & 3.5 \\ 1.9 & 0 & 0 \end{bmatrix} \qquad \mathbf{Q} = \begin{bmatrix} -5.6 & 4.7 & 0.9 \\ 0 & -3.5 & 3.5 \\ 1.9 & 0 & -1.9 \end{bmatrix}$$

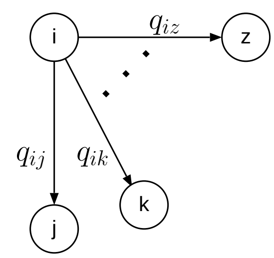
**Example 12.2**




This CTMC might represent a failure / repair model: state $D$ represents a working state, state $E$ represents a failed state, and state $F$ represents a failed state that cannot be repaired. Note that $F$ is an absorbing state.

## 12.1 Intuitive meaning of the rates

Consider a particular state $i$, with several non-zero rates of transition to other states:




To understand the meaning of these rates, let's determine

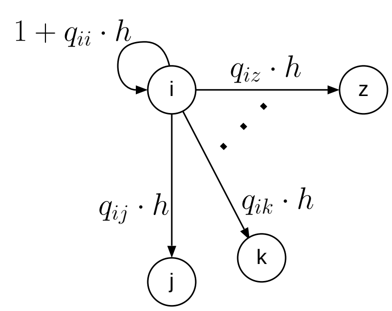
1. How long does the CTMC remain in state $i$, before changing states? This is a random variable.
2. Once leaving state $i$, which state will we enter? This is also random, so what we really want is the probability of entering each state.

### How long do we stay in state $i$

Suppose we observe the state of the CTMC every $h$ time units, for small $h$. This gives us a DTMC, where the time step for the DTMC is "observation number" of the CTMC. Since $h$ is small, we can estimate the probability of going from state $i$ to state $j$ in one observation as $q_{ij} \cdot h$, for $i \neq j$. The probability of remaining in state $i$ after one observation is

$$1 - (q_{ij} \cdot h + \cdots + q_{iz} \cdot h) \;=\; 1 + q_{ii} \cdot h$$

Thus, we obtain the following as the "observation DTMC".



Let random variable $M$ equal the time spent in state $i$, before the first state change. Let $t = n \cdot h$, i.e., the CTMC time $t$ corresponds to observation $n$ of the DTMC. What is the CDF of $M$?

$$\begin{aligned}
\Pr\{M < t\} &= \Pr\{\text{CTMC left state } i \text{ before time } t\} \\
&\approx \Pr\{\text{DTMC left state } i \text{ before observation } n\} \\
&\approx 1 - \Pr\{\text{DTMC still in state } i \text{ at observation } n\} \\
&\approx 1 - (1 + q_{ii} \cdot h)^n \\
&\approx 1 - (1 + q_{ii} \cdot t/n)^n \quad = \quad 1 - \sum_{k=0}^{n} \binom{n}{k} 1^{n-k} \cdot (q_{ii} \cdot t/n)^k
\end{aligned}$$

The last step is the binomial expansion. To obtain this probability exactly, take the limit as $n \to \infty$:

$$\begin{aligned}
\Pr\{M < t\} &= \lim_{n\to\infty} 1 - \sum_{k=0}^{n} \binom{n}{k} 1^{n-k} \cdot (q_{ii} \cdot t/n)^k \\
&= 1 - \lim_{n\to\infty} \sum_{k=0}^{n} \frac{n!}{k!(n-k)!} \, \frac{(q_{ii} \cdot t)^k}{n^k} \\
&= 1 - \lim_{n\to\infty} \sum_{k=0}^{n} \frac{n \cdot (n-1) \cdot \cdots \cdot (n-k+1)}{k!} \, \frac{(q_{ii} \cdot t)^k}{n^k} \\
& \quad \text{As } n \to \infty,\ n \cdot (n-1) \cdot \cdots \cdot (n-k+1) \to n^k \\
&= 1 - \lim_{n\to\infty} \sum_{k=0}^{n} \frac{(q_{ii} \cdot t)^k}{k!} \\
&= 1 - \sum_{k=0}^{\infty} \frac{(q_{ii} \cdot t)^k}{k!} \\
&= 1 - e^{q_{ii} \cdot t}
\end{aligned}$$

The last step is due to the Taylor expansion of $e^x$. Thus, $M \sim \mathit{Expo}(-q_{ii}) = \mathit{Expo}(q_{ij} + \cdots + q_{iz})$.

### Which state do we go to

Using the "observation DTMC", we can make all states except state $i$ absorbing, to determine what state is reached after leaving state $i$. The only transient state is $i$, and we have

$$\begin{aligned}
\mathbf{P}[\mathcal{Z}, \mathcal{Z}] &= [\ 1 + q_{ii} \cdot h\ ] \\
\mathbf{I} - \mathbf{P}[\mathcal{Z}, \mathcal{Z}] &= [\ -q_{ii} \cdot h\ ] \\
\mathbf{N} &= \left[\ -\frac{1}{q_{ii} \cdot h}\ \right] \\
\mathbf{P}[\mathcal{Z}, \mathcal{A}] &= [\ q_{ij} \cdot h\ ,\ \ldots\ ,\ q_{iz} \cdot h\ ] \\
\mathbf{N} \cdot \mathbf{P}[\mathcal{Z}, \mathcal{A}] &= \left[\ \frac{q_{ij}}{-q_{ii}}\ ,\ \ldots\ ,\ \frac{q_{iz}}{-q_{ii}}\ \right]
\end{aligned}$$

Recall that $\mathbf{N} \cdot \mathbf{P}[\mathcal{Z}, \mathcal{A}]$ gives the probability of being absorbed into each absorbing state. Now, take the limit as $h \to 0$. Since the vector does not depend on $h$, this is trivial. Thus, in the CTMC, once we leave state $i$, we go to state $j$ with probability

$$\frac{q_{ij}}{-q_{ii}} = \frac{q_{ij}}{q_{ij} + \cdots + q_{iz}}$$

## Summary of CTMC behavior

Recall: if we have

$$X_j \sim \mathit{Expo}(q_{ij}), X_k \sim \mathit{Expo}(q_{ik}), \ldots, X_z \sim \mathit{Expo}(q_{iz})$$

then $X = \min(X_j, X_k, \ldots X_z)$ has distribution $\mathit{Expo}(q_{ij} + q_{ik} + \cdots + q_{iz})$. And, $X = X_j$ with probability

$$\frac{q_{ij}}{q_{ij} + q_{ik} + \cdots + q_{iz}}$$

Therefore, we have the following.

- As soon as the CTMC enters state $i$, look at all the outgoing arcs to states $j, k, \ldots, z$ and sample a random time $X_j \sim \mathit{Expo}(q_{ij})$, $X_k \sim \mathit{Expo}(q_{ik})$, $\ldots$, $X_z \sim \mathit{Expo}(q_{iz})$ for each arc.
- Whichever time is smallest "wins", and after that amount of time, the CTMC will change to that state. E.g., if $X_m$ is the smallest, then after $X_m$ time the CTMC switches from state $i$ to state $m$.
- The CTMC will remain in state $i$ for time

  $$\min(X_j, X_k, \ldots, X_z) \sim \mathit{Expo}(-q_{ii})$$

- The CTMC will switch to state $j$ when $X_j = \min(X_j, X_k, \ldots, X_z)$, which occurs with probability

  $$\frac{q_{ij}}{q_{ij} + q_{ik} + \cdots + q_{iz}}$$